ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 5.2: Calculus of Variations

Prof. R. Hiptmair, SAM, ETH Zurich

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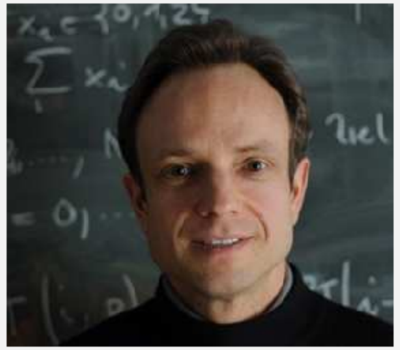

Prerequisites.
- Elementary calculus: differentiation and Taylor expansion

Dependencies. [Lecture → Section 5.1], [Lecture → Section 1.4], [Lecture → Section 1.5]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

V. Non-Linear Elliptic BVPs

5.2. Calculus of Variations

5.2.1 Calculus of Variations: Fundamental Idea

Setting: $V \hat{=}$ vector space / "hold-all" function space

$V_0 \subset V$ subspace

$\hat{V} := u_0 + V_0 \hat{=}$ affine space, $u_0 \in V$

Functional $J : \hat{V} \longrightarrow \mathbb{R}$

Global minimizer: $\boxed{u_* \in \underset{v \in \hat{V}}{\operatorname{argmin}}\, J(v)}$

▷ $\forall v \in V_0$: $\varphi_v(t) := J(u_* + tv)$ minimal in $t = 0$

Why $v \in V_0$?

(2) $\Rightarrow \quad \frac{d\varphi_v}{dt}(0) = 0 \quad \forall v \in V_0$

Theorem 5.2.1.5. Characterization of global minimizers

Let $J : \widehat{V} \to \mathbb{R}$ a continuous functional and

$$u_* \in \underset{v \in \widehat{V}}{\operatorname{argmin}}\, J(v)$$

a global minimizer of J over the affine space $\widehat{V} = v_0 + V_0 \subset V$. Then, if $\varphi_v(t) := J(u_* + tv)$ is differentiable in $t = 0$ for all $v \in V_0$,

$$\frac{d\varphi_v}{dt}(0) = 0 \quad \forall v \in V_0 \,. \tag{5.2.1.6}$$

A necessary condition

Generalizes Sect. 1.4.1, where J was a quadratic functional. ($\rightarrow$ Ex. 5.2.1.7)

Ex. 5.2.1.8: $V = \mathbb{R}^n$, $n \in \mathbb{N}$

$J : \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth

Known: $x_* \in \underset{z \in \mathbb{R}^n}{\operatorname{argmin}} J(z) \Rightarrow \operatorname{grad} J(x_*) = 0$

$\Updownarrow$

$(5.2.1.6) \Leftrightarrow \frac{d}{dt}\{t \mapsto J(x_* + tv)\} = 0\} = \operatorname{grad} J(x_*) \cdot v = 0 \quad \forall v \in \mathbb{R}^n$

How to compute $\frac{d\varphi_v}{dt}(0)$?

$\frac{d\varphi_v}{dt}(0) = \lim_{t \to 0} \frac{\varphi_v(t) - \varphi_v(0)}{t} = \lim_{t \to 0} \frac{J(u_* + tv) - J(u_*)}{t}$

$=$ directional derivative (in direction v): $\langle DJ(u_*), v \rangle$

5.2.2. Non-Linear Variational Equations

Ex. 5.2.2.1: Directional derivative for elastic string potential energy

$$J_S(u) := J_{el}(u) + J_g(u) = \int_a^b \frac{1}{2}\frac{b-a}{L}\sigma(x)\left(\sqrt{1+|u'(x)|^2} - \frac{L}{b-a}\right)^2 + g\rho(x)u(x)\,dx \,. \tag{5.1.1.33}$$

- $\langle DJ_g(u_*), v \rangle = J_g(v)$, since J_g linear [directional derivative reproduces linear functionals]
- $$J_{el}(u) = \frac{1}{2c}\int_a^b \sigma(x)\left(\sqrt{1+|u'(x)|^2} - c\right)^2 dx \,, \quad c := \frac{L}{b-a} \,. \tag{5.1.1.32}$$

DD by Taylor expansion of $t \longmapsto J_{el}(u_* + tv)$

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$$|(u_*+tv)'(x)|^2 = |u_*'(x)|^2 + 2tu_*'(x)v'(x) + O(t^2) \quad \text{for} \quad t\to 0\,,$$

$$\begin{aligned}\sqrt{1+|(u_*+tv)'(x)|^2} &= \sqrt{1+|u_*'(x)|^2 + 2tu_*'(x)v'(x) + O(t^2)}\\ &= \sqrt{1+|u_*'(x)|^2}\cdot\sqrt{1+\frac{2tu_*'(x)v'(x)}{1+|u_*'(x)|^2}+O(t^2)}\\ &= \sqrt{1+|u_*'(x)|^2}\cdot\left(1+t\frac{u_*'(x)v'(x)}{1+|u_*'(x)|^2}+O(t^2)\right) \quad \text{for} \quad t\to 0\,,\end{aligned}$$

$\sqrt{1+\delta} = 1 + \frac{1}{2}\delta + O(\delta^2)$

$$\Downarrow$$

$$\begin{aligned}\left(\sqrt{1+|(u_*+tv)'(x)|^2}-c\right)^2 &= \left(\sqrt{1+|u_*'(x)|^2}-c+t\frac{u_*'(x)v'(x)}{\sqrt{1+|u_*'(x)|^2}}+O(t^2)\right)^2\\ &= \left(\sqrt{1+|u_*'(x)|^2}-c\right)^2 + 2t\left(\sqrt{1+|u_*'(x)|^2}-c\right)\frac{u_*'(x)v'(x)}{\sqrt{1+|u_*'(x)|^2}}+O(t^2)\,.\end{aligned}$$

$$\Downarrow$$

$$J_{\mathrm{el}}(u_*+tv)-J_{\mathrm{el}}(u_*) = \frac{1}{2c}\int_a^b \sigma(x)2t\left(\sqrt{1+|u_*'(x)|^2}-c\right)\frac{u_*'(x)v'(x)}{\sqrt{1+|u_*'(x)|^2}}\,\mathrm{d}x + O(t^2)\,,$$

$$\lim_{t\to 0}\frac{J_{\mathrm{el}}(u_*+tv)-J_{\mathrm{el}}(u_*)}{t} = \frac{1}{c}\int_a^b \sigma(x)\left(\sqrt{1+|u_*'(x)|^2}-c\right)\frac{u_*'(x)v'(x)}{\sqrt{1+|u_*'(x)|^2}}\,\mathrm{d}x\,. \tag{5.2.2.4}$$

$\Rightarrow$ Variational equation solved by minimizer u_*

$$\langle \mathrm{D}J_{\mathrm{el}}(u_*),v\rangle + \langle \mathrm{D}J_{\mathrm{g}}(u_*),v\rangle = 0 \quad \forall v\in H_0^1(]a,b[)$$

$$\Updownarrow$$

$$\int_a^b \frac{1}{c}\sigma(x)\left(\sqrt{1+|u_*'(x)|^2}-c\right)\frac{u_*'(x)v'(x)}{\sqrt{1+|u_*'(x)|^2}} + g\rho(x)v(x)\,\mathrm{d}x = 0 \quad \forall v\in H_0^1(]a,b[)\,. \tag{5.2.2.5}$$

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Ex. 5.2.2.6 (Directional derivative of integral functionals)

$$J(u) = \int_\Omega F(x,\mathrm{grad}\,u,u)\,dx\,,\quad u\in C^1_{pw}(\overline{\Omega})$$

$[\,\Omega\subset\mathbb{R}^d\,,\quad F:\Omega\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}\quad C^1\text{-smooth}\,]$

Taylor expansion in multi-D

$$F(x,g+\delta g,u+\delta u) = F(x,g,u)+D_2F(x,g,u)\cdot\delta g + D_3F(x,g,u)\,\delta u$$

partial derivs.

$$\begin{aligned}J(u+tv) &= \int_\Omega F(\mathbf{x},\mathbf{grad}\,u(\mathbf{x})+t\,\mathbf{grad}\,v(\mathbf{x}),u(\mathbf{x})+tv(\mathbf{x}))\,\mathrm{d}\mathbf{x}\\ &= \int_\Omega F(\mathbf{x},\mathbf{grad}\,u(\mathbf{x}),u(\mathbf{x}))+t\Big(\mathrm{D}_2F(\mathbf{x},\mathbf{grad}\,u(\mathbf{x}),u(\mathbf{x}))\,\mathbf{grad}\,v(\mathbf{x})+\\ &\qquad \mathrm{D}_3F(\mathbf{x},\mathbf{grad}\,u(\mathbf{x}),u(\mathbf{x}))v(\mathbf{x})\Big)+O(t^2)\,\mathrm{d}\mathbf{x}\\ &= J(u)+t\int_\Omega \mathrm{D}_2F(\mathbf{x},\mathbf{grad}\,u(\mathbf{x}),u(\mathbf{x}))\,\mathbf{grad}\,v(\mathbf{x})+\\ &\qquad \mathrm{D}_3F(\mathbf{x},\mathbf{grad}\,u(\mathbf{x}),u(\mathbf{x}))v(\mathbf{x})\,\mathrm{d}\mathbf{x}+O(t^2)\,,\end{aligned}$$

$$\begin{aligned}\lim_{t\to 0}\frac{J(u+tv)-J(u)}{t} &= \langle DJ(u),v\rangle\\ &= \int_\Omega \mathrm{D}_2F(\mathbf{x},\mathbf{grad}\,u(\mathbf{x}),u(\mathbf{x}))\,\mathbf{grad}\,v(\mathbf{x})+\mathrm{D}_3F(\mathbf{x},\mathbf{grad}\,u(\mathbf{x}),u(\mathbf{x}))v(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.\end{aligned} \tag{5.2.2.10}$$

(4) Ex 5.2.2.12 (Variational equation for 2D elastic membrane)

Potential energy functional, $u \in H^1(\Omega)$

$$J_M(u) = \int_\Omega \frac{1}{2L}\sigma(\boldsymbol{x})\left(\left(\sqrt{1+\left|\frac{\partial u}{\partial x_1}(\boldsymbol{x})\right|^2} - \frac{L}{b-a}\right)^2 + \left(\sqrt{1+\left|\frac{\partial u}{\partial x_2}(\boldsymbol{x})\right|^2} - \frac{L}{b-a}\right)^2\right) + g\,\rho(\boldsymbol{x})u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,. \tag{5.1.2.14}$$

$$= \int_\Omega F(x, \operatorname{grad} u, u)\, dx$$

$$F\left(\boldsymbol{x}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, u\right) := \frac{1}{2L}\sigma(\boldsymbol{x})\left\{\left(\sqrt{1+g_1^2}-c\right)^2 + \left(\sqrt{1+g_2^2}-c\right)^2\right\} + g\,\rho(\boldsymbol{x})u\,. \tag{5.2.2.13}$$

$$\mathrm{D}_2 F\left(\boldsymbol{x}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, u\right) = \frac{1}{2L}\sigma(\boldsymbol{x})\left[\left(\sqrt{1+g_1^2}-c\right)\frac{g_1}{\sqrt{1+g_1^2}}, \left(\sqrt{1+g_2^2}-c\right)\frac{g_2}{\sqrt{1+g_2^2}}\right] \in \mathbb{R}^{1,2}\,,$$

$$\mathrm{D}_3 F\left(\boldsymbol{x}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, u\right) = g\,\rho(\boldsymbol{x}) \in \mathbb{R}\,.$$

(5.2.2.10):

$$\langle DJ_M(u), v\rangle = \int_\Omega \sigma(\boldsymbol{x})\mathbf{A}(\mathbf{grad}\, u(\boldsymbol{x}))\,\mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x}) + g\,\rho(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \tag{5.2.2.14}$$

(5.2.2.15)

$$\mathbf{A}\left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right) := \frac{1}{2L}\begin{bmatrix} \left(\sqrt{1+g_1^2}-c\right)\frac{1}{\sqrt{1+g_1^2}} & 0 \\ 0 & \left(\sqrt{1+g_2^2}-c\right)\frac{1}{\sqrt{1+g_2^2}} \end{bmatrix} \in \mathbb{R}^{2,2}$$

Elastic membrane variational equation

Find $u \in H^1(\Omega)$, $u|_{\partial\Omega} = d$, such that

$$\int_\Omega \sigma(\boldsymbol{x})\mathbf{A}(\mathbf{grad}\, u(\boldsymbol{x}))\,\mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega g\,\rho(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(\Omega)\,, \tag{5.2.2.17}$$

where $\mathbf{A}$ is defined in (5.2.2.15).

Structure of 2nd-order elliptic V.P.

§ 5.2.2.28 (General variational equations)

In all examples $v \longmapsto \langle DJ(u), v\rangle$ is linear!

Definition 5.2.2.20. (General) variational equation

An abstract (general, non-linear) **variational equation** reads

$$\boxed{u \in \widehat{V}: \quad \mathrm{a}(u;v) = 0 \quad \forall v \in V_0}\,, \tag{5.2.2.21}$$

trial space — test space

where

- $V_0 \mathrel{\hat=}$ is (real) *vector space* of functions,
- $\widehat{V} \mathrel{\hat=}$ is an *affine space* of functions: $V = u_0 + V_0$, with an offset function $u_0 \in V$,
- $\mathrm{a} \mathrel{\hat=}$ a mapping $\widehat{V} \times V_0 \mapsto \mathbb{R}$ that is *linear in the second argument v*:

$$\mathrm{a}(u; \alpha v + \beta w) = \alpha\mathrm{a}(u;v) + \beta\mathrm{a}(u;w) \quad \forall u \in V\,, v, w \in V_0\,, \alpha, \beta \in \mathbb{R}\,. \tag{5.2.2.22}$$

Ex 5.2.2.24 (Minimal surface problem for graphs)

For $u : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$G_u := \left\{ \mathbf{x} = [x_1, x_2, x_3]^\top : \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \Omega,\ x_3 = u\left(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]\right) \right\}, \tag{5.2.2.25}$$

$$\underbrace{\operatorname{area}(G_u) = \int_\Omega \sqrt{1 + \|\mathbf{grad}\, u(\mathbf{x})\|_2^2}\, d\mathbf{x}}_{J(u) :=} \longrightarrow \min \tag{5.2.2.27}$$

Directional derivative by Taylor expansion

$$\begin{aligned}
\sqrt{1 + \|\mathbf{grad}\, u(\mathbf{x}) + t\, \mathbf{grad}\, v(\mathbf{x})\|_2^2} &= \sqrt{1 + \|\mathbf{grad}\, u(\mathbf{x})\|_2^2 + 2t\, \mathbf{grad}\, u(\mathbf{x}) \cdot \mathbf{grad}\, v(\mathbf{x}) + O(t^2)} \\
&= \sqrt{1 + \|\mathbf{grad}\, u(\mathbf{x})\|_2^2} \cdot \sqrt{1 + 2t \frac{\mathbf{grad}\, u(\mathbf{x}) \cdot \mathbf{grad}\, v(\mathbf{x})}{1 + \|\mathbf{grad}\, u(\mathbf{x})\|_2^2} + O(t^2)} \\
&= \sqrt{1 + \|\mathbf{grad}\, u(\mathbf{x})\|_2^2} \left(1 + t \frac{\mathbf{grad}\, u(\mathbf{x}) \cdot \mathbf{grad}\, v(\mathbf{x})}{1 + \|\mathbf{grad}\, u(\mathbf{x})\|_2^2} + O(t^2)\right),
\end{aligned}$$

▷ Variational equation

$$\underbrace{\begin{array}{l} u \in C^1_{pw}(\Omega), \\ u|_{\partial\Omega} = d \end{array}}_{V} : \underbrace{\int_\Omega \frac{\mathbf{grad}\, u(\mathbf{x}) \cdot \mathbf{grad}\, v(\mathbf{x})}{\sqrt{1 + \|\mathbf{grad}\, u(\mathbf{x})\|_2^2}}\, d\mathbf{x}}_{= \langle DJ(u), v \rangle} = 0 \quad \forall v \in \underbrace{C^1_{pw,0}(\overline{\Omega})}_{V_0}. \tag{5.2.2.29}$$

Again: Structure of a 2nd-order elliptic V.P.

Special (simpler) cases

Ex. 5.2.2.30 (Quasi-linear 2nd-order elliptic V.P.)

$$u \in X \subset H^1(\Omega): \int_\Omega \mathbf{A}(\mathbf{x}, u(\mathbf{x}))\, \mathbf{grad}\, u(\mathbf{x}) \cdot \mathbf{grad}\, v(\mathbf{x}) + G(\mathbf{x}, u(\mathbf{x}))v(\mathbf{x})\, d\mathbf{x} = \int_\Omega f(\mathbf{x})v(\mathbf{x})\, d\mathbf{x} \quad \forall v \in X_0 \subset H^1(\Omega). \tag{5.2.2.31}$$

$A : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^{2,2}$ indep. of derivatives of u

$G : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$)

Ex. 5.2.2.32 (Semi-linear 2nd-order elliptic V.P.

$$u \in X \subset H^1(\Omega): \int_\Omega \mathbf{A}(\mathbf{x})\, \mathbf{grad}\, u(\mathbf{x}) \cdot \mathbf{grad}\, v(\mathbf{x}) + G(\mathbf{x}, u(\mathbf{x}))v(\mathbf{x})\, d\mathbf{x} = \int_\Omega f(\mathbf{x})v(\mathbf{x})\, d\mathbf{x} \quad \forall v \in X_0 \subset H^1(\Omega), \tag{5.2.2.33}$$

No derivatives in non-linear term!

Also:

$$u \in H^1(\Omega): \int_\Omega \kappa(\mathbf{x})\, \mathbf{grad}\, u \cdot \mathbf{grad}\, v\, d\mathbf{x} + \int_{\partial\Omega} \Psi(u)\, v\, dS = \int_\Omega f v\, d\mathbf{x} \quad \forall v \in H^1(\Omega), \tag{1.8.0.8}$$

5.2.3 Non-Linear BVPs

Sect. 1.5 : V.P. $\xrightarrow{\text{i.b.p.}}$ BVP

extra smoothness required

Ex 5.2.3.5 (BVP for shape of elastic membrane)

VP:

$u \in H^1(\Omega)$, $u|_{\partial\Omega} = d$:

$$\int_\Omega \sigma(x)\mathbf{A}(\mathbf{grad}\, u(x))\, \mathbf{grad}\, u(x) \cdot \mathbf{grad}\, v(x)\, \mathrm{d}x = -\int_\Omega g\,\rho(x)v(x)\, \mathrm{d}x \quad \forall v \in H_0^1(\Omega)\,, \qquad (5.2.3.6)$$

Extra smoothness: $u \in C^2$, $\sigma \in C^1$, $\rho \in C^0$

Integration by parts by

Theorem 1.5.2.7. Green's first formula

For all vector fields $\mathbf{j} \in (C^1_{\mathrm{pw}}(\overline{\Omega}))^d$ *and functions* $v \in C^1_{\mathrm{pw}}(\overline{\Omega})$ *holds*

$$\int_\Omega \mathbf{j} \cdot \mathbf{grad}\, v\, \mathrm{d}x = -\int_\Omega \operatorname{div} \mathbf{j}\, v\, \mathrm{d}x + \int_{\partial\Omega} \mathbf{j} \cdot \mathbf{n}\, v\, \mathrm{d}S\,. \qquad (5.2.3.8)$$

$\Rightarrow$

$$\int_\Omega -\operatorname{div}(\sigma(x)\mathbf{A}(\mathbf{grad}\, u(x))\, \mathbf{grad}\, u(x))v(x)\, \mathrm{d}x = -\int_\Omega g\,\rho(x)v(x)\, \mathrm{d}x \quad \forall v \in H_0^1(\Omega)\,. \qquad (5.2.3.9)$$

Lemma 1.5.3.4. Fundamental lemma of calculus of variations in higher dimensions

If $f \in L^2(\Omega)$ *satisfies*

$$\int_\Omega f(x)v(x)\, \mathrm{d}x = 0 \quad \forall v \in C_0^\infty(\Omega)\,,$$

then $f \equiv 0$ *can be concluded.*

$\Rightarrow$

$$\operatorname{div}(\sigma(x)\mathbf{A}(\mathbf{grad}\, u(x))\, \mathbf{grad}\, u(x)) = g\,\rho(x) \quad \text{in} \quad \Omega\,, \qquad (5.2.3.10)$$

$+ \quad u = d$ on $\partial\Omega$