ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 5.3: Galerkin Discretization of Non-Linear BVPS

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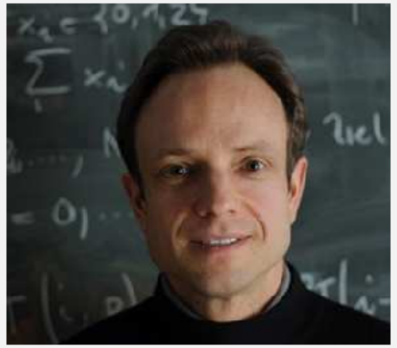

Dependencies. [Lecture → Section 5.2.2], [Lecture → Section 2.2], [Lecture → Section 2.6]

Duration: 49 minutes

⚠ Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

V. Non-Linear Elliptic BVPs

5.3 Galerkin Discretization of Non-Linear BVPs

5.3.1 Abstract Galerkin Discretization of Non-Linear Variational Problems

General variational equation (→ Def 5.2.2.20)

$$u \in \widehat{V}: \quad \mathsf{a}(u;v) = 0 \quad \forall v \in V_0 \,. \tag{5.2.2.21}$$

$\hookrightarrow \quad \widehat{V} = u_0 + V_0 \subset V$

Two-step approach as in Sect 2.2.1:

I. Replace $V_0 \longrightarrow V_{0,h} \subset V_0$, $\dim V_{0,h} =: N < \infty$

$\Rightarrow \quad \widehat{V} \longrightarrow \widehat{V}_h := u_{0,h} + V_{0,h}$

▷ Discrete variational problem:

$$u_h \in \widehat{V}_h := u_{0,h} + V_{0,h}: \quad \mathsf{a}(u_h;v_h) = 0 \quad \forall v_h \in V_{0,h} \,, \tag{5.3.1.1}$$

(2)

II. Choose ordered basis $\mathfrak{B}_h = \{b_h^j\}_{j=1}^N$ of $V_{0,h}$
$N := \dim V_{0,h}$

$\Rightarrow$ trial function: $u_h = u_{h,0} + \sum_{\ell=1}^{N} \mu_\ell b_h^\ell$, $\mu_\ell \in \mathbb{R}$

test function: $v_h \longrightarrow b_h^1, b_h^2, \dots, b_h^N$

Second step of Galerkin discretization of variational problem

discrete variational problem $\xrightarrow{\text{Choosing basis } \mathfrak{B}_h}$ (Non-linear) system of equations

$$u_h \in \widehat{V}_h: \quad \mathsf{a}(u_h; v_h) = 0 \;\; \forall v_h \in V_{0,h} \qquad \boldsymbol{F}(\vec{\boldsymbol{\mu}}) = \boldsymbol{0}\,, \tag{5.3.1.4}$$

with coefficient vector: $\vec{\boldsymbol{\mu}} = [\mu_1, \dots, \mu_h]^\top \in \mathbb{R}^N$, (5.3.1.5)

$$(\boldsymbol{F}(\vec{\boldsymbol{\mu}}))_i = \mathsf{a}(u_{o,h} + \sum_{j=1}^{N} \mu_j b_h^j; b_h^i)\,, \quad i = 1, \dots, N\,.$$

Ex. 5.3.1.6 (Lagrange FEM for 2nd-order non-linear variational equations)

$u \in C^1_{pw}(\overline{\Omega})$:

$$\mathsf{a}(u; v) := \int_\Omega D_2F(\boldsymbol{x}, \mathbf{grad}\, u(\boldsymbol{x}), u(\boldsymbol{x}))\mathbf{grad}\, v(\boldsymbol{x}) + D_3F(\boldsymbol{x}, \mathbf{grad}\, u(\boldsymbol{x}), u(\boldsymbol{x}))v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$$

$$\forall v \in C^1_{pw}(\overline{\Omega})\,, \tag{5.3.1.7}$$

$\rightarrow$ Ex 5.2.2.6

Depending on F, may be posed on "exotic" function spaces

(May not be well-defined on $H^1(\Omega)$!)

Note: Lagr. FE space $S_0^p(\mathcal{M}) \subset C^1_{pw}(\overline{\Omega})$
$\downarrow$
Almost always suitable for Gal. disc.
$\Downarrow$

$u_h \in \mathcal{S}_p^0(\mathcal{M})$:

$$\int_\Omega D_2F(\boldsymbol{x}, \mathbf{grad}\, u_h(\boldsymbol{x}), u_h(\boldsymbol{x}))\mathbf{grad}\, v_h(\boldsymbol{x}) + D_3F(\boldsymbol{x}, \mathbf{grad}\, u_h(\boldsymbol{x}), u_h(\boldsymbol{x}))v_h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$$

$$\forall v_h \in \mathcal{S}_p^0(\mathcal{M})\,, \tag{5.3.1.8}$$

[Discrete variational problem]

5.3.2 Iterative Methods in Function Space

Option: Solve NLSE $F(\vec{\mu}) = 0$ by some iterative method.

Alternative: Idea: Design the iterative method for (5.2.2.21) before discretization!

The usual warnings apply

In many cases iterative method will only converge, if the initial guess is chosen *"sufficiently" close* to the (desired) solution.

Solutions of (general) discrete variational problems need *not* be *unique*: the (approximate) solution computed by a convergent iterative method may depend on the intial guess.

5.3.2.1 Fixed-Point Iterations

Model problem, $\Omega \subset \mathbb{R}^d$:

$$u \in \hat{V} \subset H^1(\Omega): \quad \int_\Omega \mathbf{A}(\boldsymbol{x}, \mathbf{grad}\, u(\boldsymbol{x}))\, \mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x}) + c(u(\boldsymbol{x}))u(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \forall v \in V_0 \subset H^1(\Omega).$$

$\mathbf{A}: \Omega \times \mathbb{R}^d \to \mathbb{R}^{d,d}$, $c: \mathbb{R} \to \mathbb{R}$

Examples:

(i) Elastic membrane V.P.

Find $u \in H^1(\Omega)$, $u|_{\partial\Omega} = d$, such that

$$\int_\Omega \sigma(\boldsymbol{x})\mathbf{A}(\mathbf{grad}\, u(\boldsymbol{x}))\, \mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = -\int_\Omega g\, \rho(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(\Omega), \tag{5.2.2.17}$$

where $\mathbf{A}$ is defined in (5.2.2.15).

(ii) Graph minimal surface V.P.

$$u \in C^1_{\mathrm{pw}}(\Omega),\ u|_{\partial\Omega} = d: \quad \int_\Omega \frac{1}{1 + \|\mathbf{grad}\, u(\boldsymbol{x})\|_2^2}\, \mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0 \quad \forall v \in C^1_{\mathrm{pw},0}(\overline{\Omega}). \tag{5.2.2.29}$$

Idea: $u^{(k)} \in \hat{V}$

- Iterate: $u^{(0)} \to u^{(1)} \to u^{(2)} \to \dots$
- To compute $u^{(n)}$: replace $u \leftarrow u^{(n-1)}$ until linear V.P.

▷ 1-point iteration:

Linearizing fixed-point iteration for (5.3.2.1)

Given $u^{(k)}$ compute $u^{(k+1)}$ as solution of the *linear* variational equation

$$u^{(k+1)} \in \hat{V} \subset H^1(\Omega): \quad \int_\Omega \mathbf{A}(\boldsymbol{x}, \mathbf{grad}\, u^{(k)}(\boldsymbol{x}))\, \mathbf{grad}\, u^{(k+1)}(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x}) + c(u^{(k)}(\boldsymbol{x}))u^{(k+1)}(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \forall v \in V_0 \subset H^1(\Omega). \tag{5.3.2.3}$$

5.3.2.2. Newton's Method

Recall: Newton's Method in $\mathbb{R}^N$ for $F(\underline{x}) = 0$

Idea: Local linearization & iteration

$$F(\underline{x}^{(k+1)}) \approx F(\underline{x}^{(k)}) + DF(\underline{x}^{(k)})(\underline{x}^{(k+1)} - \underline{x}^{(k)}) \overset{!}{=} 0$$

$\Rightarrow$ Newton iteration:

$$\begin{cases} DF(\underline{x}^{(k)})\,\underline{w} = -F(\underline{x}^{(k)}) \\ \underline{x}^{(k+1)} = \underline{x}^{(k)} + \underline{w} \end{cases}, \quad k \in \mathbb{N}_0$$

How to adapt this idea to

$$\boxed{u \in \widehat{V} := u_0 + V_0\colon \quad \mathsf{a}(u;v) = 0 \quad \forall v \in V_0} \quad , \qquad ? \qquad (5.2.2.21)$$

Linearization:

$$a(u^{(k+1)};v) = a(u^{(k)};v) + D_u a(u^{(k)};v)(u^{(k+1)} - u^{(k)}) \overset{!}{=} 0 \quad \forall v \in V_0$$

What is $D_u a(u,v)w$? Jacobian of $a(\cdot;\cdot)$?

Hint: For $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$

$$DF(\underline{x})\underline{h} = \lim_{t \to 0} \frac{F(\underline{x} + t\underline{h}) - F(\underline{x})}{t} \quad \forall \underline{h} \in \mathbb{R}^N$$

$\hat{=}$ a directional derivative

$$D_u a(u;v)w := \lim_{t \to 0} \frac{a(u + tw;v) - a(u;v)}{t}$$

$\hookrightarrow$ still linear in $v \in V_0$

As d.d: Also linear in $w \in V_0$

$\Rightarrow (w,v) \longmapsto D_u a(u;v)w$ is a bilinear form on V_0

$\triangleright$ Newton iteration for (5.2.2.21):

$$\begin{cases} w \in V_0\colon \quad \mathsf{D}_u\mathsf{a}(u^{(k)};v)w = -\mathsf{a}(u^{(k)};v) \quad \forall v \in V_0, & (5.3.2.9) \\ u^{(k+1)} = u^{(k)} + w \end{cases}$$

(5.3.2.9) $\hat{=}$ linear variational problem

Ex. 5.3.2.10 ($D_u a(\cdot;\cdot)$ for semi-linear V.P.)

$$\mathsf{a}(u;v) := \underbrace{\int_\Omega \kappa(x)\,\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,\mathrm{d}x}_{=:\mathsf{b}(u,v)} + \underbrace{\int_{\partial\Omega}\Psi(u)\,v\,\mathrm{d}S}_{=:\mathsf{c}(u;v)}\,,\quad u,v\in X\subset H^1(\Omega)\,. \tag{5.3.2.11}$$

→ Ex. 1.8.0.6 & Ex. 5.2.2.32

Obvious: $D_u a(u;v)w = D_u b(u,v)w + D_u c(u;v)w$

(i) Easy: b bilinear: $D_u b(u,v)w = b(w,v)$!

(ii)

$$\mathsf{c}(u;v) := \int_{\partial\Omega}\Psi(u)v\,\mathrm{d}S\,,\quad u,v\in X\,.$$

$$\mathsf{c}(u+tw;v)-\mathsf{c}(u;v) = \int_{\partial\Omega}(\Psi(u+tw)-\Psi(u))v\,\mathrm{d}S\,,\quad u,v\in X\,.$$

Tool: Taylor expansion

$$\mathsf{c}(u+tw;v)-\mathsf{c}(u;v) = \int_{\partial\Omega} t\Psi'(u)wv\,\mathrm{d}S + O(t^2)\,.$$

$$\mathsf{D}_u\mathsf{c}(u;v)w = \lim_{t\to 0}\frac{\mathsf{c}(u+tw;v)-\mathsf{b}(u;v)}{t} = \int_{\partial\Omega}\Psi'(u)w\,v\,\mathrm{d}S\,.$$

↔ Sect. 5.2.2 (Derivation of variational equations) !

5.3.3 Galerkin Discretization of Linearized Variational Equations

Benefit of "iterations in function space":

Step-dependent (adaptive) choice of Galerkin trial/test spaces.

For iterative methods in function space different Galerkin trial/test spaces can be chosen differently in every step.

§ 5.3.3.2 (Galerkin discretization for fixed-point method)

$u^{(k+1)}\in\widehat{V}\subset H^1(\Omega)$:

$$\int_\Omega \mathbf{A}(x,\mathbf{grad}\,u^{(k)}(x))\,\mathbf{grad}\,u^{(k+1)}(x)\cdot\mathbf{grad}\,v(x) + c(u^{(k)}(x))u^{(k+1)}(x)v(x)\,\mathrm{d}x$$
$$= \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \forall v\in V_0\subset H^1(\Omega)\,. \tag{5.3.2.3}$$

Discrete test space $V_{0,h}^{(k)} := S_p^0(\mathcal{M}^{(k)})$ (Lagrange FE space)

Discrete trial space $\widehat{V}_h^{(k)} := u_{0,h}^{(k)} + V_{0,h}^{(k)}$

offset function

⑥ Choose bases $\mathcal{B}_h^{(k)} := \{b_k^1, \dots, b_k^{N_k}\}$ of $V_{0,h}^{(k)}$, $N_k := \dim V_{0,h}^{(k)}$

▷ Iterates $\vec{\mu}^{(k)} \in \mathbb{R}^{N_k}$ from

$$\mathbf{M}(\vec{\mu}^{(k-1)})\vec{\mu}^{(k)} = \vec{\varphi}(\vec{\mu}^{(k-1)}) \tag{5.3.3.4}$$

with

$$u_h^{(k)} = u_{0,h}^{(k)} + \sum_{\ell=1}^{N_k} \left(\vec{\mu}^{(k)}\right)_\ell b_k^\ell \quad , \quad u_h^{(k-1)} = u_{0,h}^{(k-1)} + \sum_{\ell=1}^{N_{k-1}} \left(\vec{\mu}^{(k-1)}\right)_\ell b_{k-1}^\ell \, ,$$

$$\mathbf{M}(\vec{\mu}^{(k-1)}) := \left[\int_\Omega \mathbf{A}(\boldsymbol{x}, \mathbf{grad}\, u_h^{(k-1)}(\boldsymbol{x}))\, \mathbf{grad}\, b_k^j(\boldsymbol{x}) \cdot \mathbf{grad}\, b_k^i(\boldsymbol{x}) + c(u_h^{(k-1)}) b_k^j(\boldsymbol{x})\, b_k^i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right]_{i,j=1}^{N_k} \in \mathbb{R}^{N_k,N_k} \, ,$$

$$\boldsymbol{\varphi}(\vec{\mu}^{(k-1)}) := \left[\int_\Omega f(\boldsymbol{x}) b_k^i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{A}(\boldsymbol{x}, \mathbf{grad}\, u_h^{(k-1)}(\boldsymbol{x}))\, \mathbf{grad}\, u_{0,h}^{(k)}(\boldsymbol{x}) \cdot \mathbf{grad}\, b_k^i(\boldsymbol{x}) + c(u_h^{(k-1)}) u_{0,h}^{(k)}(\boldsymbol{x})\, b_k^i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right]_{i=1}^{N_k} \in \mathbb{R}^{N_k} \, .$$

§ 5.3.3.5 (Galerkin discretization for Newton's method)

V.P.: $\boxed{u \in \widehat{V} := u_0 + V_0: \quad \mathsf{a}(u;v) = 0 \quad \forall v \in V_0}$, (5.2.2.21)

▷ Newton iteration for (5.2.2.21):

$$\begin{cases} w \in V_0: \quad \mathsf{D}_u \mathsf{a}(u^{(k)};v)w = -\mathsf{a}(u^{(k)};v) \quad \forall v \in V_0 \, , & (5.3.2.9) \\ u^{(k+1)} = u^{(k)} + w \end{cases}$$

Galerkin test space : $V_{0,h}^{(k)} \subset V_0$

Galerkin trial space : $\hat{V}_h^{(k)} := u_{0,h}^{(k)} + V_{0,h}^{(k)}$

▷ Discrete linear variational equation for Newton update

$$w_h \in V_{0,h}^{(k)}: \quad \mathsf{D}_u \mathsf{a}(u_h^{(k-1)};v_h)w_h = -\mathsf{a}(u_h^{(k-1)};v_h) \quad \forall v_h \in V_{0,h}^{(k)} \, , \tag{5.3.3.6}$$

+ Computation of next ~~iterate~~ → projection/interpolation

$$u_h^{(k)} := \mathsf{P}_h^{(k)}(u_h^{(k-1)} + w_h) \, . \quad \left[u_h^{(k-1)} + w_h \notin \hat{V}_h^{(k)} \text{ possible} \right]$$

7 Choose bases $\mathcal{B}_h^{(k)} := \{b_k^1, \dots, b_k^{N_k}\}$ of $V_{0,h}^{(k)}$, $N_k := \dim V_{0,h}^{(k)}$

$\Rightarrow$ Newton updates $\vec{\omega} \in \mathbb{R}^{N_k}$

LSE :

$$\mathbf{M}(\vec{\mu}^{(k-1)})\vec{\omega} = \vec{\varphi}(\vec{\mu}^{(k-1)}) , \tag{5.3.3.7}$$

with

$$w_h = \sum_{\ell=1}^{N_k} (\vec{\omega})_\ell b_k^\ell , \quad u_h^{(k-1)} = u_{0,h}^{(k-1)} + \sum_{\ell=1}^{N_{k-1}} \left(\vec{\mu}^{(k-1)}\right)_\ell b_{k-1}^\ell ,$$

$$\mathbf{M}(\vec{\mu}^{(k-1)}) := \left[\mathsf{D}_u \mathsf{a}(u_h^{(k-1)}; b_k^i) b_k^j\right]_{i,j=1}^{N_k} \in \mathbb{R}^{N_k, N_k} , \quad \vec{\varphi}(\vec{\mu}^{(k-1)}) := \left[-\mathsf{a}(u_h^{(k-1)}; b_k^i)\right]_{i=1}^{N_k} \in \mathbb{R}^{N_k} .$$

$\Rightarrow$ Next iterate $\mu^{(k)} \in \mathbb{R}^{N_k}$

$$\sum_{\ell=1}^{N_k} \left(\vec{\mu}^{(k)}\right)_\ell b_k^\ell = \mathsf{P}_h^{(k)}(u_h^{(k-1)} + w_h) - u_{0,h}^{(k)} . \tag{5.3.3.8}$$

Answer Review Questions 5.3.3.10 without aids!