ETH Lecture 401-0674-00L Numerical Methods for ~~Partial~~ Differential Equations

Course Video

Section 6.1: Initial-Value Problems (IVPs) for Ordinary Differential Equations (ODEs)

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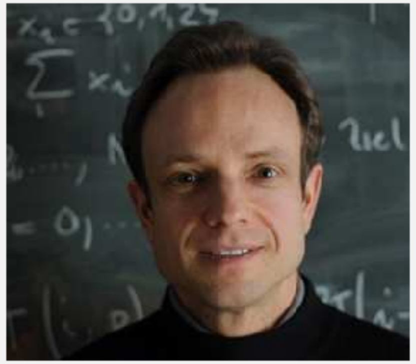

Date: March 27, 2021

Prerequisites.

- Concept of ODE from analysis course
- Basic results about existence and uniqueness of solutions of initial-value problems

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

VI. Numerical Integration – Single-Step Methods

6.1. Initial-Value Problems for ODEs

6.1.1. Ordinary Differential Equations (ODEs)

$$\frac{dy}{dt}(t) =: \dot{y}(t) = f(t, y(t)) \qquad \text{(ODE)}$$

$f : I \times D \to D$: r.h.s. function, continuous

$\uparrow$ interval

$D \subset \mathbb{R}^N \;\hat{=}\;$ state space (configuration space)

$y \in C^1(I, D)$ is a solution, if (ODE) satisfied $\forall t \in I$

For $N > 1$ $\quad \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ can be viewed as a system of ordinary differential equations:

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \quad \Longleftrightarrow \quad \begin{bmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_N \end{bmatrix} = \begin{bmatrix} f_1(t, y_1, \ldots, y_N) \\ \vdots \\ f_N(t, y_1, \ldots, y_N) \end{bmatrix}.$$

2 Smoothness of solutions:

$y = y(t)$ is a solution of $\dot{y} = f(y)$ & $f \in C^m(D)$

$\Rightarrow \quad y \in C^{m+1}(I, D)$

Linear ODEs

Scalar case, $N=1$: $\quad \dot{y} = a y$, $a \in \mathbb{R}$

One-parameter family of solution: $\quad y(t) = c e^{at}$, $c \in \mathbb{R}$

$N > 1$: $\quad \dot{y} = A y$, $A \in \mathbb{R}^{N,N}$

Diagonalization:

$\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$, $\mathbf{S} \in \mathbb{R}^{N,N}$ regular, $\mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^{N,N}$, (6.1.1.13)

$$\dot{\mathbf{y}} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}\mathbf{y} \Rightarrow \dot{\mathbf{z}} = \mathbf{D}\mathbf{z} \text{ with } \mathbf{z}(t) := \mathbf{S}^{-1}\mathbf{y}(t).$$

Decoupled scalar ODEs: $\dot{z}_j = \lambda_j z_j$

$$\mathbf{y}(t) = \mathbf{S}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{bmatrix}\mathbf{S}^{-1}\mathbf{w} \quad \text{for some } \mathbf{w} \in \mathbb{R}^N. \tag{6.1.1.14}$$

parameter

N-parameter family of solution

6.1.2. Mathematical Modeling with ODEs

- Classical rigid-body mechanics: $\ddot{y} = f(y)$ [Newton's law]
- Electrical circuits, transients
- Chemical reaction kinetics

Ex. 6.1.2.5 (Population dynamics: predator-prey model)

$N=2$: $y(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ $\begin{matrix} \to \text{prey} \\ \to \text{predator} \end{matrix}$ (population densities) [viewed as continuum vars]

State space $D = [0, \infty[^2 \subset \mathbb{R}^2$

ODE-based model: autonomous Lotka-Volterra ODE:

$$\begin{aligned} \dot{u} &= (\alpha - \beta v)u \\ \dot{v} &= (\delta u - \gamma)v \end{aligned} \leftrightarrow \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) \text{ with } \mathbf{y} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{f}(\mathbf{y}) = \begin{bmatrix} (\alpha - \beta v)u \\ (\delta u - \gamma)v \end{bmatrix}, \tag{6.1.2.6}$$

with positive model parameters $\alpha, \beta, \gamma, \delta > 0$.

$v = 0$: $\quad \dot{u} = \alpha u \Rightarrow$ exponential growth

$u = 0$: $\quad \dot{v} = -\gamma v \Rightarrow$ exp. decay

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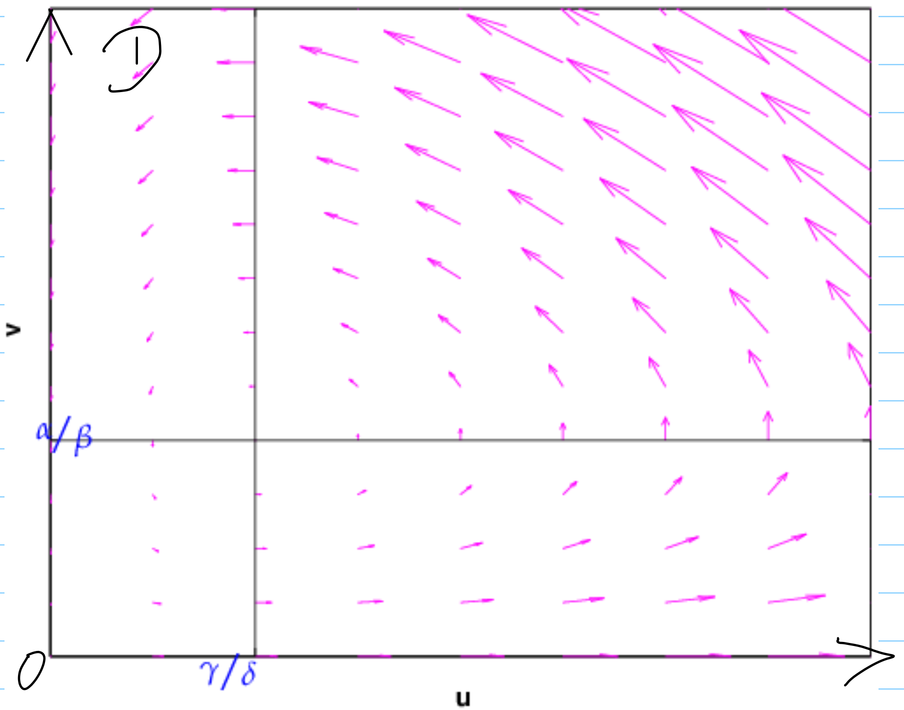


≙ state plane

$\rightarrow$ ≙ $y \rightarrow f(y)$

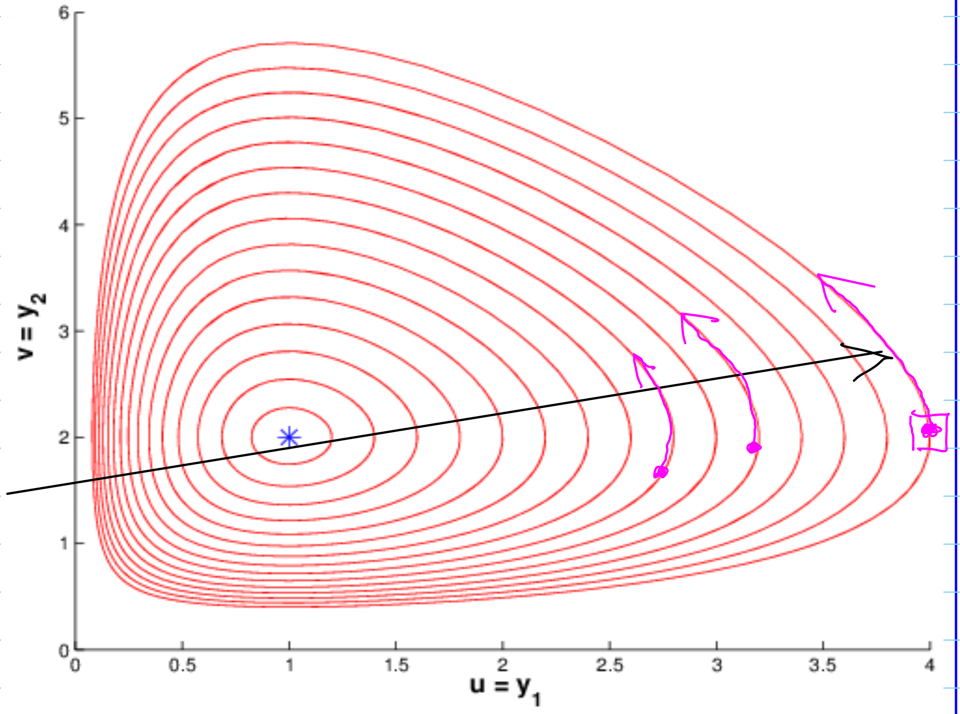
f ≙ vector field
velocity field

$t \mapsto y(t) \in \mathbb{R}^2$

≙ a curve in the plane

A solution curve

a trajectory



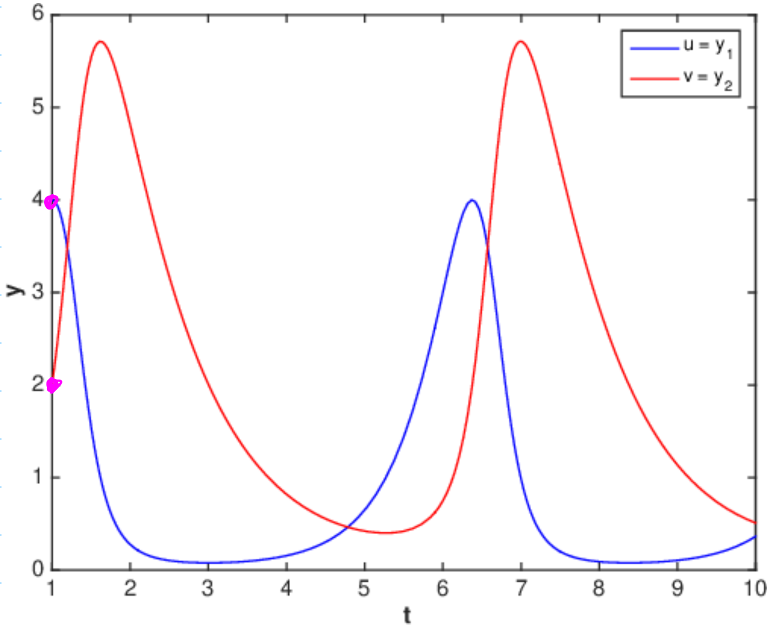
Solution curves for (6.1.2.6)

Need to impose an initial state at some initial time t_0 to obtain a unique solution

$\Downarrow$

IVP

$\dot{y} = f(y)$ & $y(t_0) = y_0$



Solution $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ for $\mathbf{y}_0 := \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Definition 6.1.2.4. Autonomous ODE

An ODE of the form $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, that is, with a right hand side function that does not depend on time, but only on state, is called **autonomous**.

§ 6.1.3.3 (IVPs for autonomous ODEs)

$t \rightarrow y(t)$ solves $\dot{y} = f(y)$

$\Rightarrow$ $t \mapsto y(t - \tau)$, $\tau \in \mathbb{R}$, will also be a solution

$\Rightarrow$ We can always fix $t_0 = 0$!

6.1.3. Theory of initial-value problems (IVPs)

A generic **Initial value problem** (IVP) for a first-order ordinary differential equation (ODE) can be stated as: find a function $\mathbf{y} : I \to D$ that satisfies, *cf.* Def. 6.1.1.2,

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \quad , \quad \mathbf{y}(t_0) = \mathbf{y}_0 \,. \tag{6.1.3.2}$$

- $\mathbf{f} : I \times D \mapsto \mathbb{R}^N \triangleq$ right hand side (r.h.s.) $(N \in \mathbb{N})$,
- $I \subset \mathbb{R} \triangleq$ (time)interval $\leftrightarrow$ "time variable" t,
- $D \subset \mathbb{R}^N \triangleq$ state space/phase space $\leftrightarrow$ "state variable" $\mathbf{y}$,
- $\Omega := I \times D \triangleq$ extended state space (of tupels $(t, \mathbf{y})$),
- $t_0 \in I \triangleq$ initial time, $\mathbf{y}_0 \in D \triangleq$ initial state $\succ$ **initial conditions.**

mild smoothness requirement, satisfied for $f \in C^1$

Theorem 6.1.3.16. Theorem of Peano & Picard-Lindelöf , , ,

If the right hand side function $\mathbf{f} : \hat{\Omega} \mapsto \mathbb{R}^N$ is locally Lipschitz continuous ($\to$ Def. 6.1.3.12) then for all initial conditions $(t_0, \mathbf{y}_0) \in \hat{\Omega}$ the IVP

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \quad , \quad \mathbf{y}(t_0) = \mathbf{y}_0 \,. \tag{6.1.3.2}$$

has a solution $\mathbf{y} \in C^1(J(t_0, \mathbf{y}_0), \mathbb{R}^N)$ with maximal (temporal) domain of definition $J(t_0, \mathbf{y}_0) \subset \mathbb{R}$.

existence & uniqueness

intrinsic temporal domain of definition

Ex. 6.1.3.19 ("Explosion equation")

Scalar IVP: $\dot{y} = y^2, \quad y(0) = y_0 \in D := \mathbb{R}$

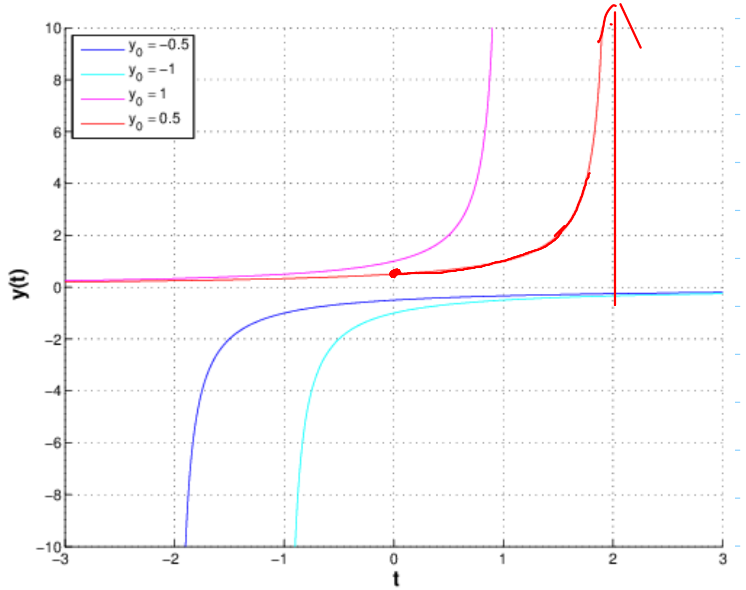
We find the solutions (a family)

$$y(t) = \begin{cases} \dfrac{1}{y_0^{-1} - t} & , \text{ if } y_0 \neq 0 \,, \\ 0 & , \text{ if } y_0 = 0 \,, \end{cases} \tag{6.1.3.21}$$

with domains of definition

$$J(y_0) = \begin{cases}]-\infty, y_0^{-1}[& , \text{ if } y_0 > 0 \,, \\ \mathbb{R} & , \text{ if } y_0 = 0 \,, \\ ]y_0^{-1}, \infty[& , \text{ if } y_0 < 0 \,. \end{cases}$$

finite-time blow-up



§ 6.1.4.3. (Autonomization: Conversion into autonomous ODE)

Definition 6.1.2.4. Autonomous ODE

An ODE of the from $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, that is, with a right hand side function that does not depend on time, but only on state, is called **autonomous**.

$$\dot{y} = f(t, y) \; : \quad z(t) = \begin{bmatrix} y(t) \\ t \end{bmatrix} \in D \times I \subset \mathbb{R}^{N+1}$$

$$\dot{z}(t) = \begin{bmatrix} \dot{y}(t) \\ 1 \end{bmatrix} = \begin{bmatrix} f(z_{N+1}, z_1, \dots, z_N) \\ 1 \end{bmatrix} \triangleq \text{Aut ODE in } z$$

⑤

Rem 6.1.3.5: From higher-order ODEs to first-order ODEs

2nd-order ODE (autonomous): $\ddot{y} = f(y)$

New unknown $z(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \in \mathbb{R}^{2N}$

$$\dot{z}(t) = \begin{bmatrix} \dot{y}(t) \\ f(y) \end{bmatrix} = \begin{bmatrix} z_{N+1} \\ \vdots \\ z_{2N} \\ f(z_1,\dots,z_N) \end{bmatrix}$$

+ initial values $z(0) = z_0 \Leftrightarrow \begin{cases} y(0) = y_0 \in \mathbb{R}^N \\ \dot{y}(0) = v_0 \in \mathbb{R}^N \end{cases}$

For ODEs of order $n \in \mathbb{N}$ well-posed initial value problems need to specify initial values for the first $n-1$ derivatives.

Focus on 1st-order autonomous IVPs

$$\dot{y} = f(y), \quad y(0) = y_0 \in \mathbb{R}^N$$

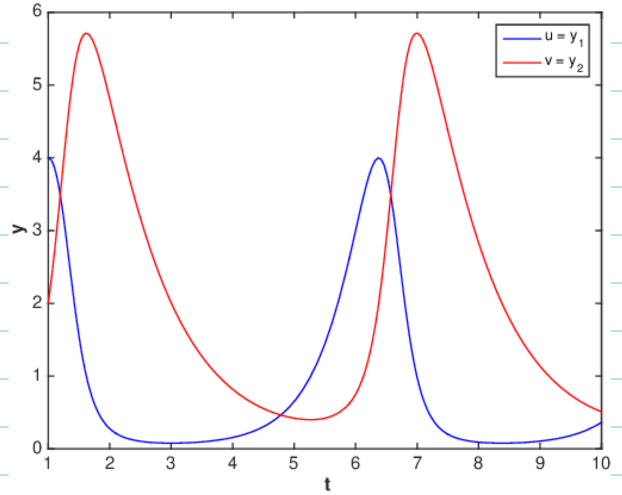
6.1.4. Evolution Operators

IVP: $\dot{y} = f(y)$, $y(0) = y_0 \in D \subset \mathbb{R}^N$ (6.1.3.17)

$f \in C^1$

Assumption 6.1.4.1. Global solutions (for simplicity)

Solutions of (6.1.3.17) are global: $J(\mathbf{y}_0) = \mathbb{R}$ for *all* $\mathbf{y}_0 \in D$.



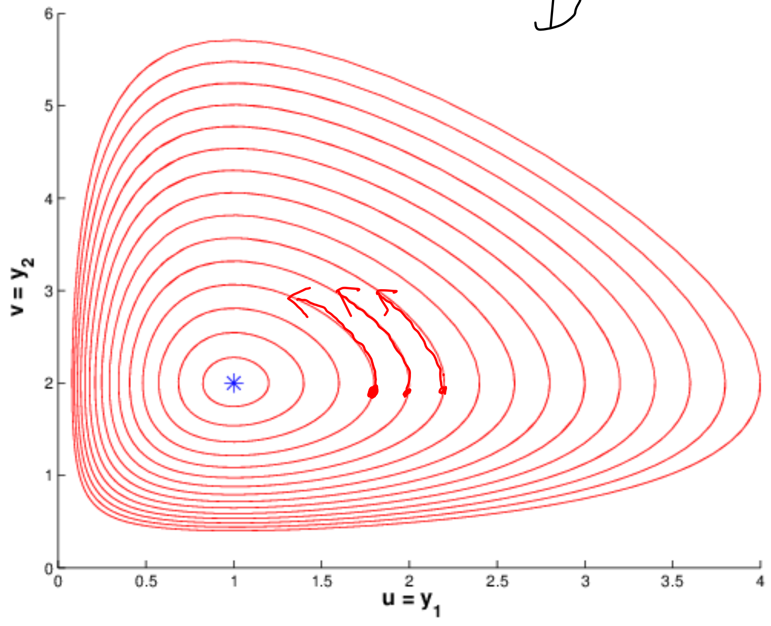
Solution $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ for $\mathbf{y}_0 := \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Two perspective

I. y_0 fixed

Trajectory $t \longmapsto y(t)$

D



Solution curves for (6.1.2.6)

II. Time t fixed

$y_0 \longrightarrow y(t)$

solution of IVP

$D \longrightarrow D$

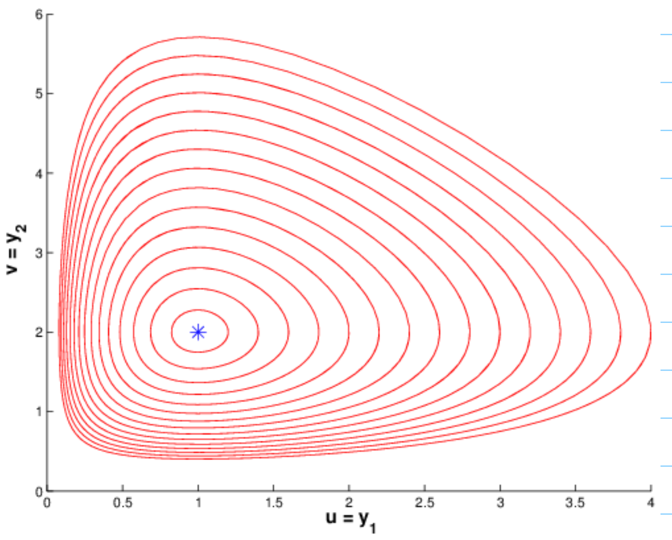
Definition 6.1.4.3. Evolution operator/mapping

Under Ass. 6.1.4.1 the mapping

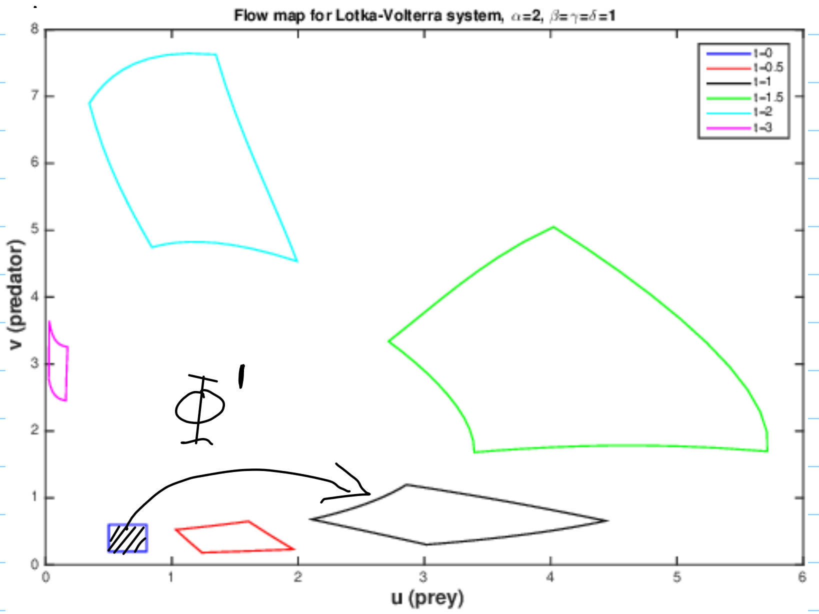
$$\mathbf{\Phi} : \begin{cases} \mathbb{R} \times D & \mapsto D \\ (t, \mathbf{y}_0) & \mapsto \mathbf{\Phi}^t \mathbf{y}_0 := \mathbf{y}(t) \end{cases},$$

where $t \mapsto \mathbf{y}(t) \in C^1(\mathbb{R}, \mathbb{R}^N)$ is the unique (global) solution of the IVP $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$, is the **evolution operator**/mapping for the autonomous ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$.

Illustration for Ex. 6.1.2.5 :



trajectories $t \mapsto \mathbf{\Phi}^t \mathbf{y}_0$



state mapping $\mathbf{y} \mapsto \mathbf{\Phi}^t \mathbf{y}$

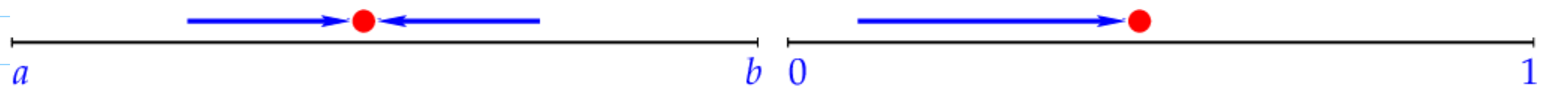
- $f(y) = \frac{\partial \Phi}{\partial t}(0, y) \Rightarrow \Phi$ contains ODE
- Autonomous ODEs : $\Phi^s \circ \Phi^t = \Phi^{s+t}$ (group property)

Numerical integration is concerned with the approximation of evolution operators.

Rem. 6.1.3.22 (IVPs vs. BVPs)

2nd-order scalar elliptic 2-pt. BVP on $[a,b]$:

$$-\frac{d}{dx}\left(\sigma(x)\frac{du}{dx}(x)\right) = f \quad \text{in} \quad]a,b[\quad , \quad u(a) = u_a \, , \quad u(b) = u_b \, . \tag{1.5.1.16}$$



Please answer review questions

After you have watched the video take a short break and then try to answer the

review questions 6.1.4.8

Please do not flip pages of the lecture document, nor look at tablet notes.