ETH Lecture 401-0674-00L Numerical Methods for [redacted] Differential Equations

# Course Video

## Section 6.2: Introduction: Polygonal Approximation Methods


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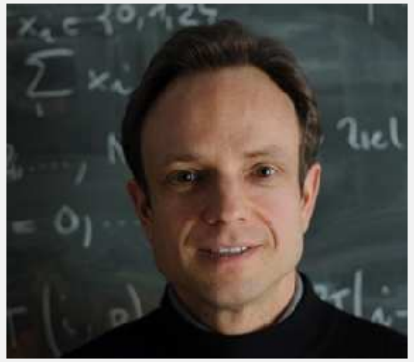


**Prerequisites.**

- Familiarity with numerical differentiation by difference quotients

**Dependencies.** [Lecture → Section 6.1.1] and [Lecture → Section 6.1.3]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]



# VI. Numerical Integration – Single-Step Methods

## 6.2. Introduction: Polygonal Approximation Methods

IVP: $\dot{\mathbf{y}} = \mathbf{f}(t,\mathbf{y})$ , $\mathbf{y}(t_0) = \mathbf{y}_0 \in D \subset \mathbb{R}^N$

given only procedural form (as function object)

```
Eigen::VectorXd operator () (double t, const Eigen::VectorXd &y)
  const;
```

Goal: Approximate $t \longmapsto \mathbf{y}(t)$ on a temporal mesh (time grid)

$$\mathcal{M} := \{t_0 < t_1 < t_2 < \cdots < t_{M-1} < t_M := T\} \subset [t_0, T] \,, \qquad (6.2.0.3)$$

final time

Autonomous IVP: $t_0 = 0$

② 

## 6.2.1. Explicit Euler Method

$$\dot{y} = f(t,y)$$

Difference quotient on $\mathcal{M}$

$$\frac{y(t_{k+1}) - y(t_k)}{h_k} \approx f(t_k, y(t_k))$$

$[\, h_k := t_{k+1} - t_k \,]$

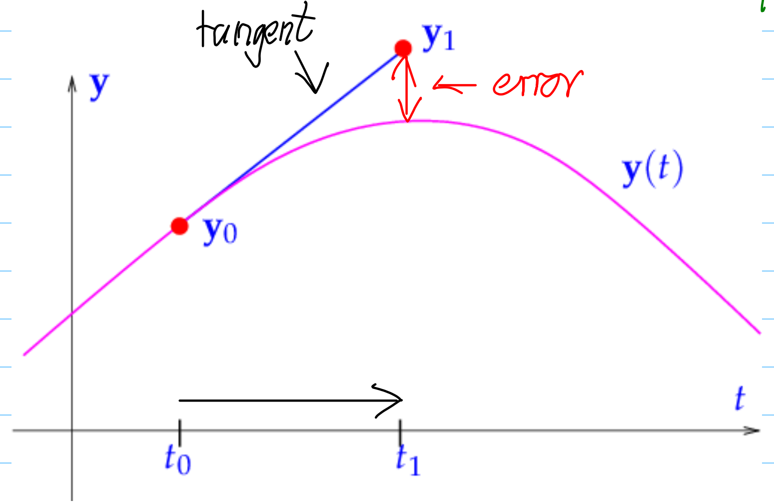
$$\Downarrow$$

Expl. Euler : $\quad y_{k+1} - y_k = h_k f(t_k, y_k)$

▷ Generates sequence $(y_k)_{k=0}^{M} \subset D$, $y_k \approx y(t_k)$

$y_{k+1}$ from $y_k$ by f-evaluations only : "explicit"

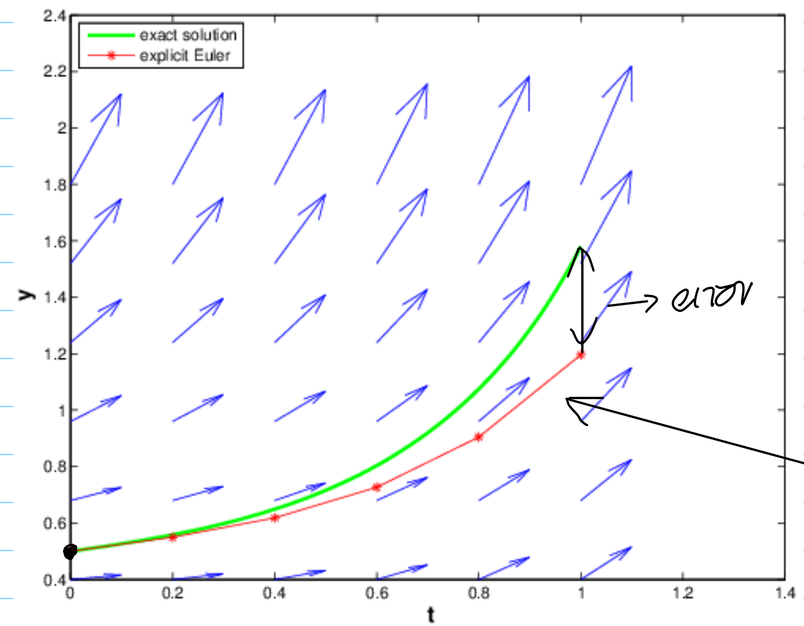
Ex. 6.2.1.3 : (Visualization of explicit Euler method)




$N = 1$ :

$$\dot{y} = f(t,y)$$

slope of tangent !




Ricatti ODE, $N=1$

$$\dot{y} = t^2 + y^2$$

$\longrightarrow \;\hat{=}$ tangent field

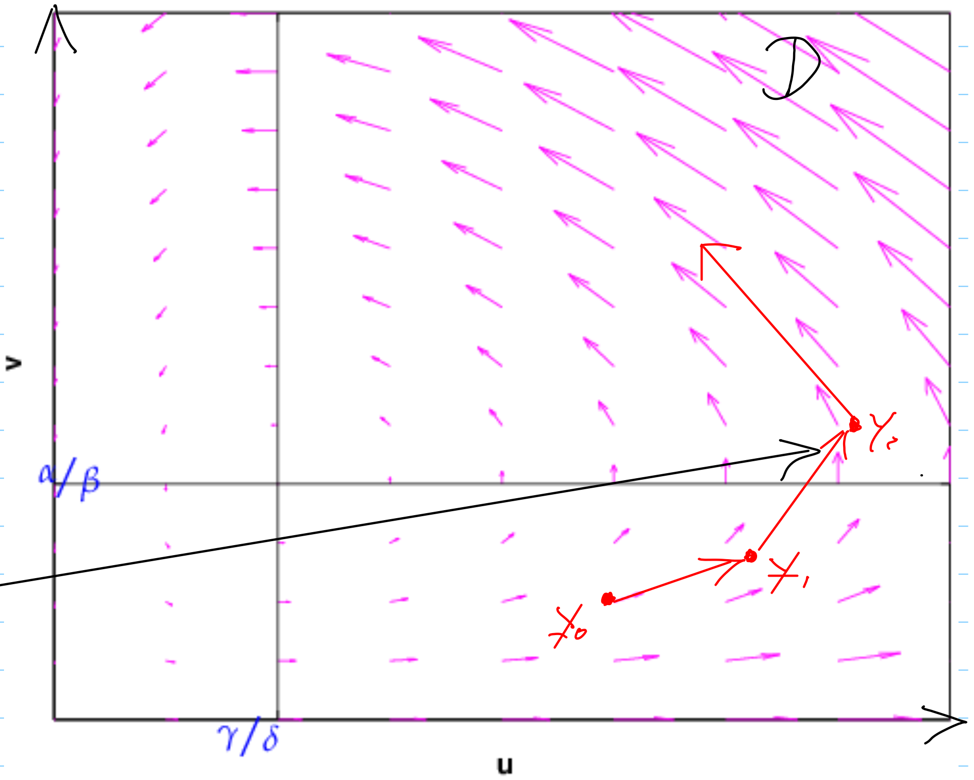
Euler polygon

Lotka-Volterra ODE:

$\longrightarrow \;\hat{=}$ velocity vector field

(equidistant $\mathcal{M}$)

Euler polygon in state space



③

# 6.2.2. Implicit Euler Method

Difference quotient on $\mathcal{M}$

$[h_k := t_{k+1} - t_k]$

$$\dot{y} = f(t,y)$$

$$\frac{y(t_{k+1}) - y(t_k)}{h_k} \approx f(t_{k+1}, y(t_{k+1}))$$

$$\Downarrow$$

Now: backward difference quotient

$$y_{k+1} - y_k = h_k f(t_{k+1}, y_{k+1})$$

▷ Need to solve (system of) equations to compute $y_{k+1}$ from $y_k$

$$\boxed{\mathbf{y}_{k+1} = \mathbf{y}_k + h_k \mathbf{f}(t_{k+1}, \mathbf{y}_{k+1})\,, \quad k = 0, \ldots, M-1}\,, \tag{6.2.2.2}$$

with local timestep (stepsize) $h_k := t_{k+1} - t_k$ .

(6.2.2.2) = implicit Euler method

## Rem 6.2.2.3: Feasibility* of implicit Euler timestepping

* (6.2.2.2) has a solution $y_{k+1}$

Intuition: $h_k = 0 \Rightarrow$ "unique solution" $y_{k+1} = y_k$

"Continuity argument": For $h_k \approx 0$ unique solution should still exist.

$\mathbf{y}_{k+1} = \mathbf{y}_k + h_k \mathbf{f}(t_{k+1}, \mathbf{y}_{k+1}) \Leftrightarrow \mathbf{G}(h, \mathbf{y}_{k+1}) = 0$ with $\mathbf{G}(h, \mathbf{z}) := \mathbf{z} - h\mathbf{f}(t_{k+1}, \mathbf{z}) - \mathbf{y}_k$ .

parameter; system of equation

$$\frac{d\mathbf{G}}{d\mathbf{z}}(h, \mathbf{z}) = \mathbf{I} - h\mathrm{D}_{\mathbf{y}}\mathbf{f}(t_{k+1}, \mathbf{z}) \Rightarrow \frac{d\mathbf{G}}{d\mathbf{z}}(0, \mathbf{z}) = \mathbf{I}\,.$$

▷ (Implicit function theorem)

For sufficiently small $|h|$: ex. & unique of $y_{k+1}$

4

# 6.2.3 Implicit midpoint method

Again : replace $\dot{y} \rightarrow$ difference quotient on $M$

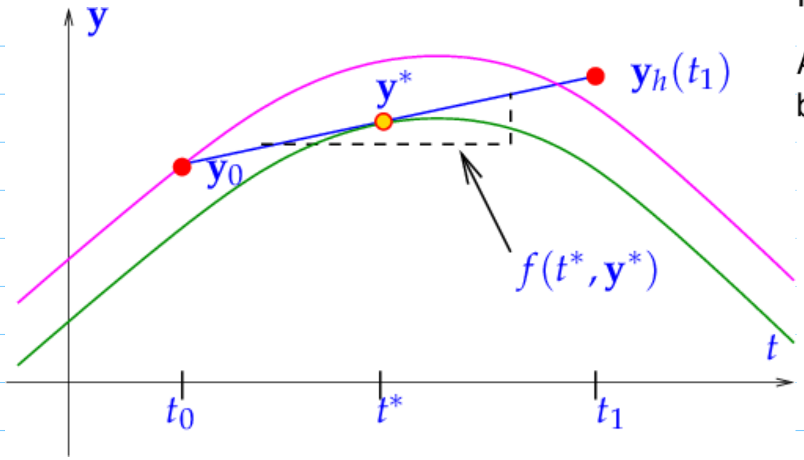
$$\dot{y} = f(t, y)$$

$$\Downarrow$$

$$\frac{y_{k+1} - y_k}{h_k} \approx f\left(\tfrac{1}{2}(t_k + t_{k+1}), \tfrac{1}{2}(y_k + y_{k+1})\right)$$

$$\Downarrow$$

$y_{k+1}$ : $y_{k+1} = y_k + h_k f\left(\tfrac{1}{2}(t_k + t_{k+1}), \tfrac{1}{2}(y_k + y_{k+1})\right)$




Implicit midpoint method, a geometric view:

Approximaate trajectory through $(t_0, \mathbf{y}_0)$ on $[t_0, t_1]$ by

- straight line through $(t_0, \mathbf{y}_0)$
- with slope $f(t^*, \mathbf{y}^*)$, where $t^* := \frac{1}{2}(t_0 + t_1)$, $\mathbf{y}^* = \frac{1}{2}(\mathbf{y}_0 + \mathbf{y}_1)$

◁ — $\hat{=}$ trajectory through $(t_0, y_0)$,
— $\hat{=}$ trajectory through $(t^*, y^*)$,
— $\hat{=}$ tangent at — in $(t^*, y^*)$.

**Please answer review questions**

After you have watched the video take a short break and then try to answer the

**review questions 6.2.3.4**

Please avoid consulting the lecture document or tablet notes.