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ETH Lecture 401-0674-00L Numerical Methods for ~~Partial~~ Differential Equations

Course Video

Section 6.3.2: (Asymptotic) Convergence of Single-Step Methods

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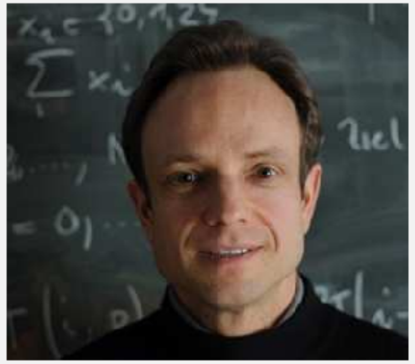


Dependencies. [Lecture → Section 3.2.2], [Lecture → Section 6.1.4], [Lecture → Section 6.3.1]

Duration: minutes

⚠ Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

VI. Numerical Integration – Single-Step Methods

6.3 General Single-Step Methods

6.3.1 (Asymptotic) Convergence of SSMs

IVP: $\dot{y} = f(y)$, $y(0) = y_0 \in D \subset \mathbb{R}^N$

SSM: $y_{k+1} = \Psi^{h_k} y_k$, $h_k := t_{k+1} - t_k$

on mesh $\mathcal{M} = \{0 = t_0 < t_1 < \ldots < t_M = T_0\}$

Focus: h-convergence, consider family of temporal meshes with $h_\mathcal{M} := \max_k h_k \longrightarrow 0$

Discretization error:

- $\max_{k=1}^{M} \| y_k - y(t_k) \|$ $\quad [t \mapsto y(t) \,\hat{=}\, \text{solution}]$
- $\| y_M - y(T) \|$

2

Exp. 6.3.2.5 : (Speed of convergence of polygonal methods)

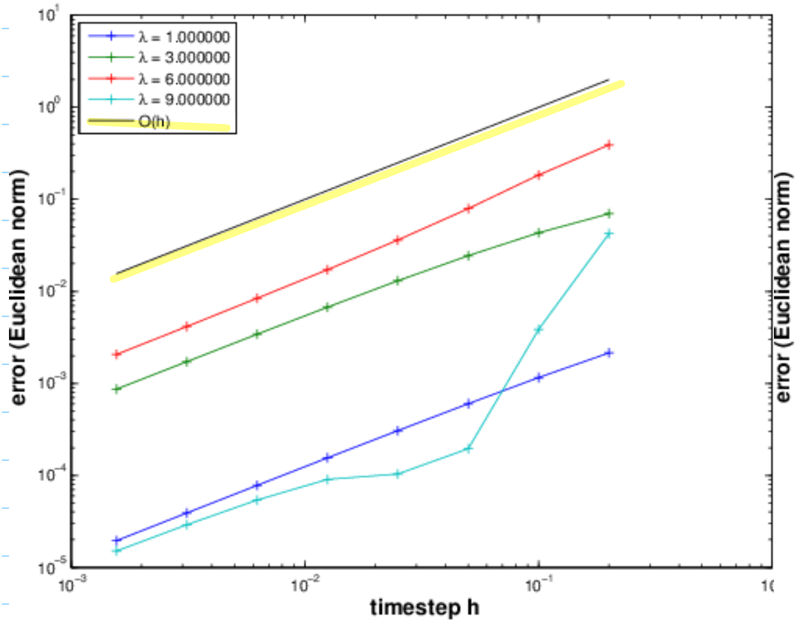
- We consider the following IVP for the logistic ODE, see Ex. 6.1.2.1

$$\dot{y} = \lambda y(1-y) \quad , \quad y(0) = 0.01 \, .$$

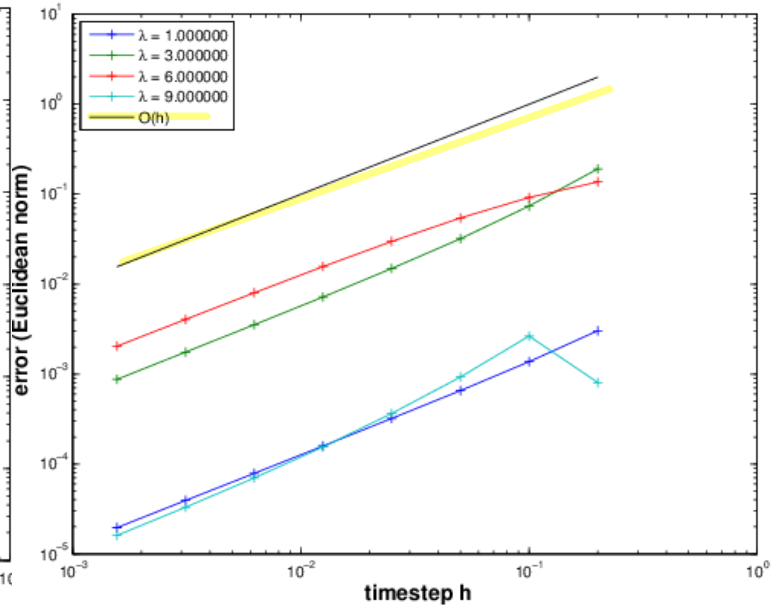
- We apply explicit and implicit Euler methods (6.2.1.4)/(6.2.2.2) with uniform timestep $h = 1/M$, $M \in \{5, 10, 20, 40, 80, 160, 320, 640\}$.

equidistant $\mathcal{M}$

- Monitored: Error at final time $E(h) := |y(1) - y_M|$



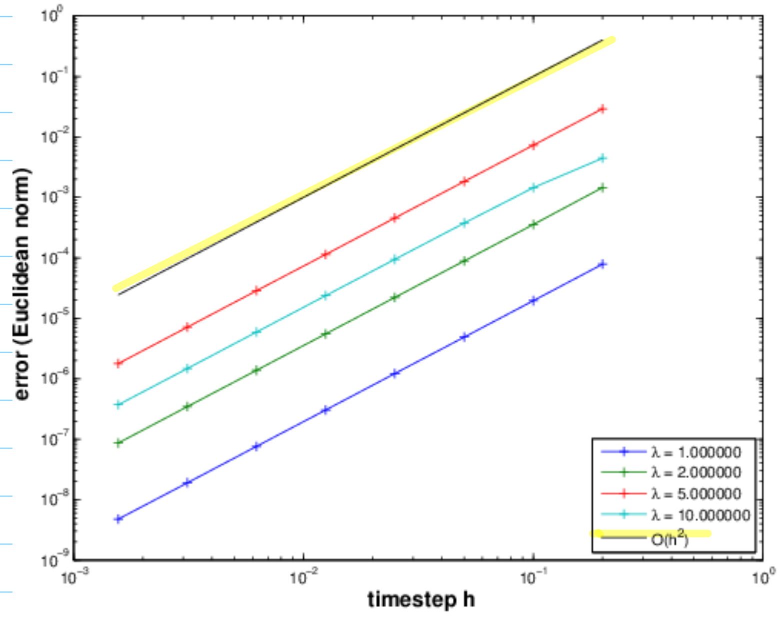
explicit Euler method



implicit Euler method

▷ Empiric algebraic cvg. : $\mathcal{E}(h) = O(h) = O(M^{-1})$

[rate/order = 1] for $h \to 0$, $M \to \infty$



implicit midpoint method

$\triangleleft$ $\mathcal{E}(h) = O(h^2)$

Alg. cvg., rate/order = 2

Algebraic convergence of single step methods

Consider the numerical integration of an initial value problem

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \quad , \quad \mathbf{y}(t_0) = \mathbf{y}_0 \, , \tag{6.1.3.2}$$

with sufficiently smooth right hand side function $\mathbf{f} : I \times D \to \mathbb{R}^N$.

Then conventional single step methods (→ Def. 6.3.1.4) will enjoy *algebraic convergence in the meshwidth*, more precisely, see ,

> there is a $p \in \mathbb{N}$ such that the sequence $(\mathbf{y}_k)_k$ generated by the single step method for $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ on a mesh $\mathcal{M} := \{t_0 < t_1 < \cdots < t_M = T\}$ satisfies
>
> $$\max_k \|\mathbf{y}_k - \mathbf{y}(t_k)\| \leq Ch^p \quad \text{for} \quad h := \max_{k=1,\dots,M} |t_k - t_{k-1}| \to 0 \, , \tag{6.3.2.7}$$
>
> with $C > 0$ independent of $\mathcal{M}$

usually unknown

③

Definition 6.3.2.8. Order of a single step method

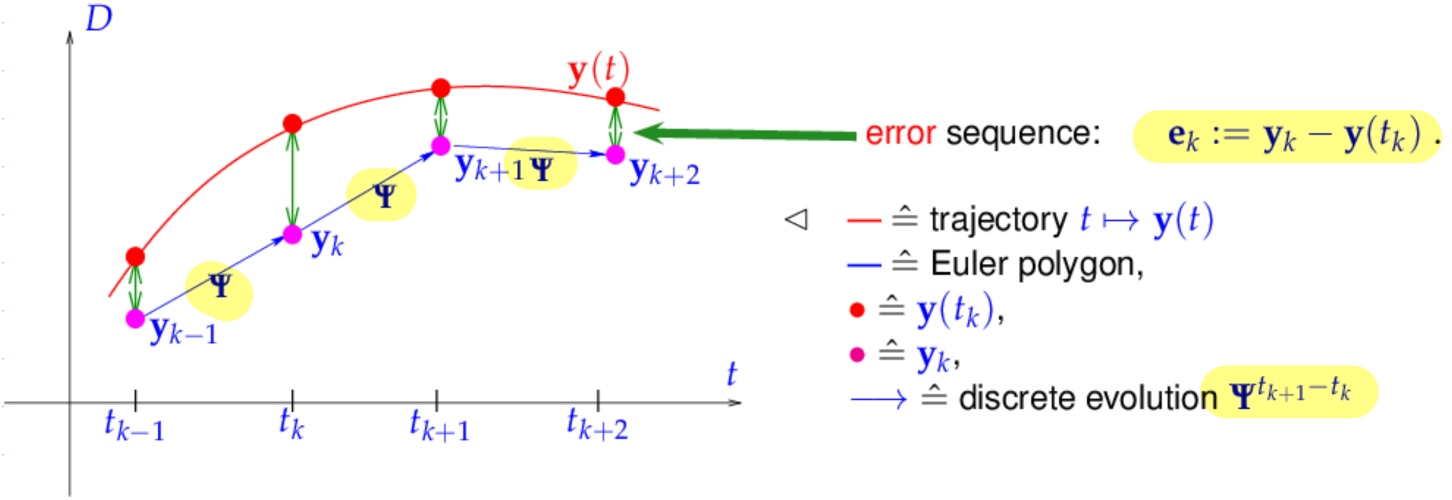
The maximal integer $p \in \mathbb{N}$ for which (6.3.2.7) holds for a single step method when applied to an ODE with (sufficiently) smooth right hand side, is called the **order** of the method.

§ 6.3.2.9 Convergence analysis for the explicit Euler method

- IVP : $\dot{y} = f(y)$, $y(0) = y_0 \in D$
- Global Lipschitz condition : (L.C.)

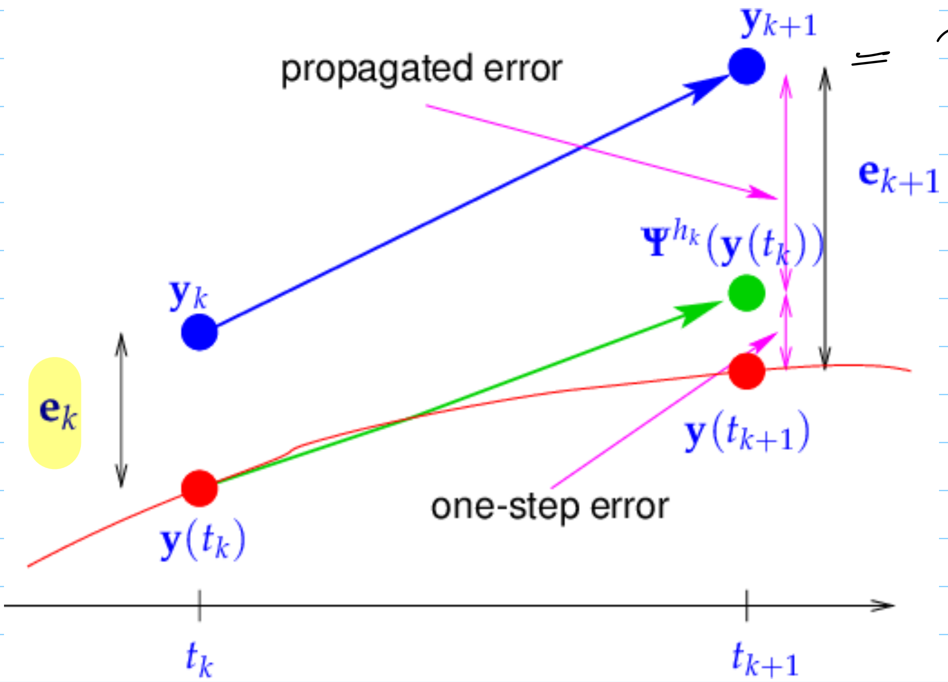
$$\exists L > 0: \quad \|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z})\| \leq L\|\mathbf{y} - \mathbf{z}\| \quad \forall \mathbf{y}, \mathbf{z} \in D\,, \tag{6.3.2.10}$$

Expl. Euler: $\mathbf{y}_{k+1} = \mathbf{y}_k + h_k \mathbf{f}(\mathbf{y}_k)\,, \quad k = 1, \ldots, M-1\,.$ (6.2.1.4)



Fundamental error splitting:

$$\mathbf{e}_{k+1} = \overbrace{\Psi^{h_k}\mathbf{y}_k}^{y_{k+1}} - \overbrace{\Phi^{h_k}\mathbf{y}(t_k)}^{y(t_{k+1})}$$
$$= \underbrace{\Psi^{h_k}\mathbf{y}_k - \Psi^{h_k}\mathbf{y}(t_k)}_{\text{propagated error}} \tag{6.3.2.12}$$
$$+ \underbrace{\Psi^{h_k}\mathbf{y}(t_k) - \Phi^{h_k}\mathbf{y}(t_k)}_{\text{one-step error}}\,.$$

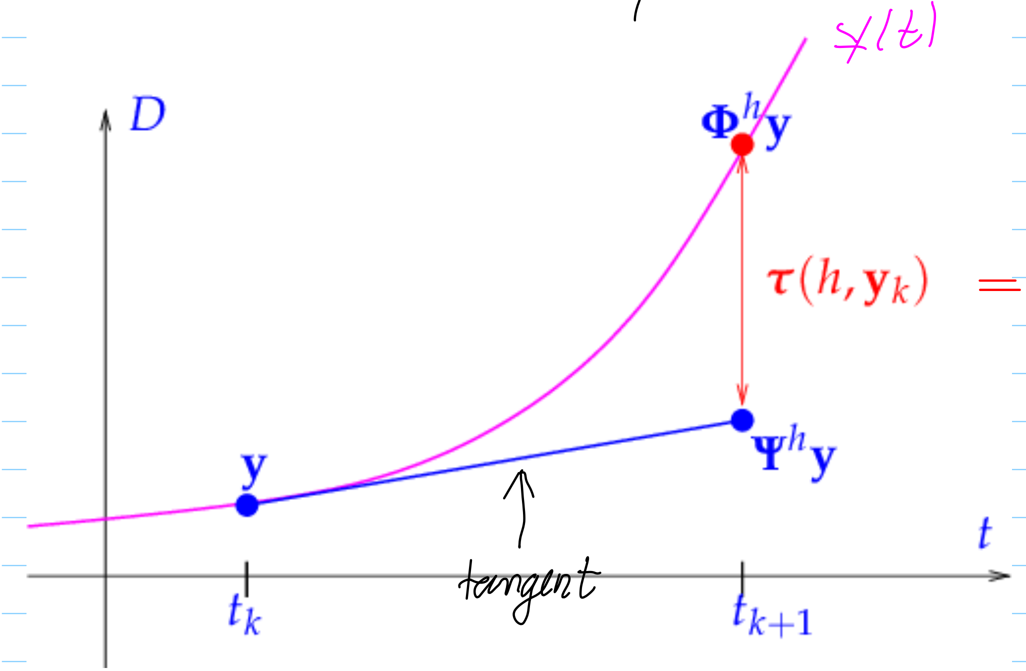


$= \Psi^h y_k$

Goal : recursion for e_k

(4)

- Estimate for one-step error:



$= O(h^2)$ (*)(*)

for $h \to 0$

- Estimate for propagated error:

$$\|\Psi^h y_k - \Psi^h y(t_k)\| = \|y_k + hf(y_k) - y(t_k) - hf(y(t_k))\|$$

L.C.

$$\underbrace{\leq (1+hL)\|y_k - y(t_k)\|}_{+\ (6.3.2.12)}$$

$$[\epsilon_k := \|e_k\|]: \quad \epsilon_{k+1} \leq (1+hL)\,\epsilon_k + \|\tau(h_k, y_k)\|$$

$$(*) \quad \epsilon_{k+1} \leq (1+hL)\,\epsilon_k + O(h^2) \Rightarrow \epsilon_k = O(h)$$

Total error arises from accumulation of propagated one-step errors!

True for all consistent SSMs

Order of algebraic convergence of single-step methods

Consider an IVP (6.1.3.2) with solution $t \mapsto \mathbf{y}(t)$ and a single step method defined by the discrete evolution $\boldsymbol{\Psi}$ ($\rightarrow$ Def. 6.3.1.4). If the *one-step error along the solution trajectory* satisfies ($\boldsymbol{\Phi}$ is the evolution map associated with the ODE, see Def. 6.1.4.3)

$$\left\|\boldsymbol{\Psi}^h \mathbf{y}(t) - \boldsymbol{\Phi}^h \mathbf{y}(t)\right\| \leq C h^{p+1} \quad \forall h \text{ sufficiently small, } t \in [0, T]\,, \tag{6.3.2.22}$$

one-step error

for some $p \in \mathbb{N}$ and $C > 0$, then, usually,

$$\max_k \|\mathbf{y}_k - \mathbf{y}(t_k)\| \leq \overline{C} h_{\mathcal{M}}^p\,,$$

with $\overline{C} > 0$ independent of the temporal mesh $\mathcal{M}$: The (pointwise) discretization error *converges algebraically* with order/rate p.

Ex. 6.3.2.25 (Order of finite-difference SSMs)

Implicit midpoint method (IMM)

$$\Psi^\tau y := w(\tau) \; : \; w(\tau) = y + \tau f(\tfrac{1}{2}(y + w(\tau)))$$

Tool: Taylor expansions

$$\Phi^\tau y = y + \frac{\partial \phi}{\partial t}(0, y)\tau + \frac{1}{2}\frac{\partial^2 \phi}{\partial \tau^2}(0, y)\tau^2 + O(\tau^3)$$
$$= y + f(y)\tau + \tfrac{1}{2}\tau^2 Df(y) f(y) + O(\tau^3)$$

by chain rule

$$w(\tau) = y + \tau f(y + \tfrac{1}{2}\tau f(\tfrac{1}{2}(y + w(\tau))))$$

Then Taylor expansion of f around y

$$\triangleright \quad w(\tau) = y + \tau f(y) + \tfrac{1}{2}\tau^2 Df(y) f(y) + O(\tau^3)$$

$$\triangleright \quad \Phi^\tau y - \Psi^\tau y = O(\tau^3)$$

IMM has order 2

Please answer review questions

After you have watched the video take a short break and then try to answer the

review questions 6.3.2.33

Please do not flip pages of the lecture document, nor look at tablet notes.