ETH Lecture 401-0674-00L Numerical Methods for ~~Partial~~ Differential Equations

Course Video
Section 6.3: General Single-Step Methods

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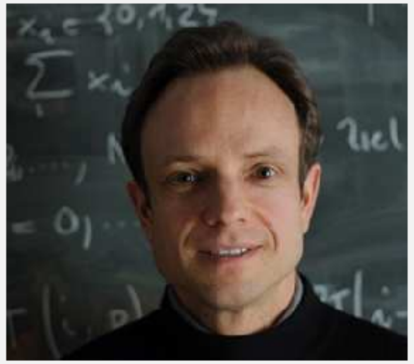

Dependencies. [Lecture → Section 6.1.4]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

VI. Numerical Integration – Single-Step Methods

6.3 General Single-Step Methods

6.3.1 Definition

Wlog: restriction to autonomous ODEs $\dot{y} = f(y)$

6.2.1: given temporal mesh $\mathcal{M} = \{0 = t_0 < t_1 < \dots < t_M\}$,

explicit Euler produces sequence of states

$$y_0, y_1, y_2, \dots, y_M$$

by $y_{k+1} := y_k + h_k f(y_k)$, $k = 0, \dots, M-1$

More general: $y_{k+1} = \Psi(h_k, y_k) =: \Psi^{h_k} y_k$

with mapping $\Psi : I \times D \to \mathbb{R}^N$, $I =]-\delta, \delta[$

Expl. Euler: $\Psi^{\tau} y = y + \tau f(y_k)$

(2)

§ 6.3.1.1 Discrete evolution operators

Expl. Euler : $y_{k+1} = \Psi^{h_k} y_k$

$t \to y(t) \hat{=}$ solution of IVP w/ $y(0) = y_0$: $y(t_{k+1}) = \Phi^{h_k} y(t_k)$

evolution operator $\Phi : I \times D \to \mathbb{R}^N$

$\triangleright$ $\Psi \hat{=}$ discrete evolution operator

$\uparrow$ supposed to approximate Φ : $\Psi \approx \Phi$

Numerical integration is concerned with the approximation of evolution operators.

Definition 6.3.1.4. Single step method (for autonomous ODE) $\rightarrow$

Given a discrete evolution $\boldsymbol{\Psi} : \Omega \subset \mathbb{R} \times D \mapsto \mathbb{R}^N$, an initial state $\mathbf{y}_0$, and a temporal mesh $\mathcal{M} := \{0 =: t_0 < t_1 < \cdots < t_M := T\}$, $M \in \mathbb{N}$, the recursion

$$\mathbf{y}_{k+1} := \boldsymbol{\Psi}(t_{k+1} - t_k, \mathbf{y}_k) , \quad k = 0, \ldots, M-1 , \tag{6.3.1.5}$$

defines a **single-step method** (SSM) for the autonomous IVP $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$ on the interval $[0, T]$.

Rem : Ψ may be "implicit"

e.g. implicit Euler : $\Psi^h y := z$: $z = y + hf(y)$

§ 6.3.1.7 Consistent SSMs

$\Psi \approx \Phi$? Recall $\frac{d}{dt} \Phi^t y |_{t=0} = f(y)$

Want $\underbrace{\frac{d}{dh} \Psi^h y |_{h=0} = f(y)}$

A minimal requirement consistency

Consistent discrete evolution

The discrete evolution $\boldsymbol{\Psi}$ defining a single step method according to Def. 6.3.1.4 and (6.3.1.5) for the autonomous ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ invariably is of the form

$$\boldsymbol{\Psi}^h \mathbf{y} = \mathbf{y} + h\boldsymbol{\psi}(h, \mathbf{y}) \quad \text{with} \quad \begin{array}{l} \boldsymbol{\psi} : I \times D \to \mathbb{R}^N \text{ continuous,} \\ \boldsymbol{\psi}(0, \mathbf{y}) = \mathbf{f}(\mathbf{y}) . \end{array} \tag{6.3.1.10}$$

$\frac{d}{dh} \Psi^h y |_{h=0} = \psi(h,y) + h \cdot \frac{\partial \psi}{\partial h}(h,y) |_{h=0}$

$= \psi(0,y) = f(y)$

II Expl. Euler : $\psi(h,y) = f(y)$

Ex. 6.3.1.11 : Consistency of implicit midpoint method

Assume : $\underline{f}$ smooth

$$y_{k+1} = y_k + h f(y_k + \tfrac{1}{2}(y_{k+1} - y_k))$$

$$= y_k + h f(y_k + \tfrac{1}{2} h f(y_k + \tfrac{1}{2}(y_{k+1} - y_k)))$$

Trick: Taylor exp. of f around y_k
[Assume h is small]

$$= y_k + h \underbrace{\left(\underline{f}(y_k) + \tfrac{1}{2} h\, D\underline{f}(y_k)\underline{f}(\ldots) + O(h^2)\right)}_{=\Psi(h, y_k)}$$

$$\Psi(0, y_k) = f(y_k) \quad \checkmark$$

§ 6.3.1.14 : Output of SSMs

$\triangleq$ $\underbrace{\text{sequence of states } (y_k)_{k=0}^{M}}$, but no trajectory ?

Idea: interpolate $\Rightarrow$ function $[0, T] \longrightarrow D$

$$\dot{y}(t_k) = \underline{f}(y(t_k)) \approx \underline{f}(y_k)$$

use for interpolation

$\Rightarrow$ Option : p.w. cubic Hermite interpolation

Please answer review questions

After you have watched the video take a short break and then try to answer the

review questions 6.3.1.16

You should do this without consulting other parts of the lecture document or tablet notes.