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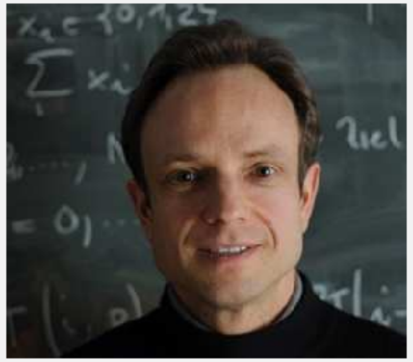
ETH Lecture 401-0674-00L Numerical Methods for ~~Partial~~ Differential Equations

Course Video
Section 7.2: Stiff Initial-Value Problems



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Date: April 5, 2021

Dependencies. [Lecture → Section 6.1], [Lecture → Section 7.1]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

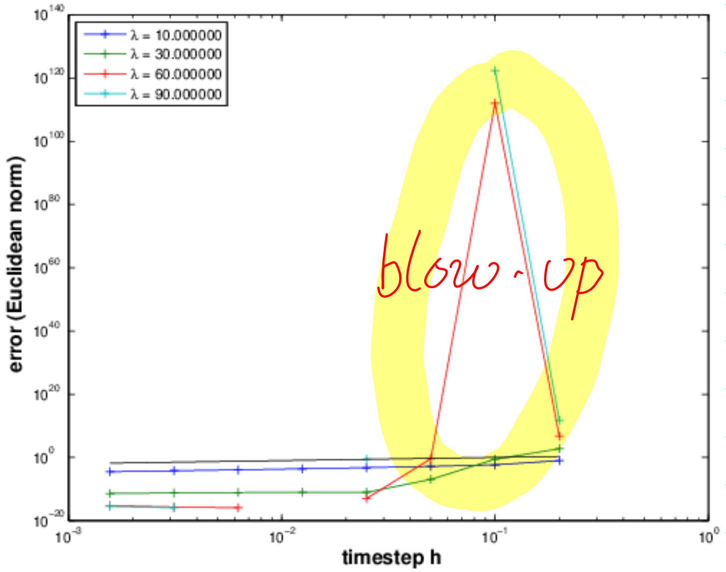
VII. Single-Step Methods for Stiff Initial-Value Problems

7.2 Stiff Initial-Value Problems

Sect. 7.1: Linear model problem analysis for explicit RK-SSMs

→ Quantification of stability-induced timestep constraints for (linear) IVPs with decaying solutions.

Ex. 7.1.0.3 (Blow-up of explicit Euler method)



Logistic ODE

$\dot{y} = \lambda y(1-y)$

$y(0) = 0.01$

◁ Convergence of expl. Euler

λ large: blow-up of y_k for large timestep h

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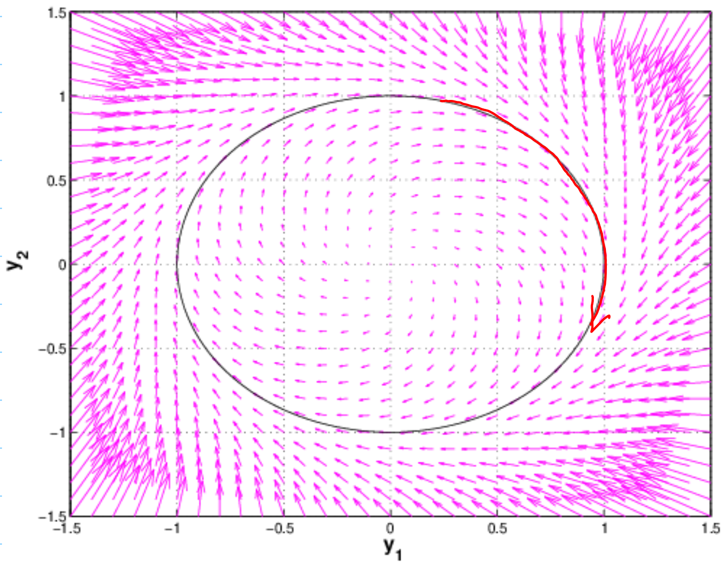
Ex. 7.2.0.4 (Strongly attractive limit cycle)

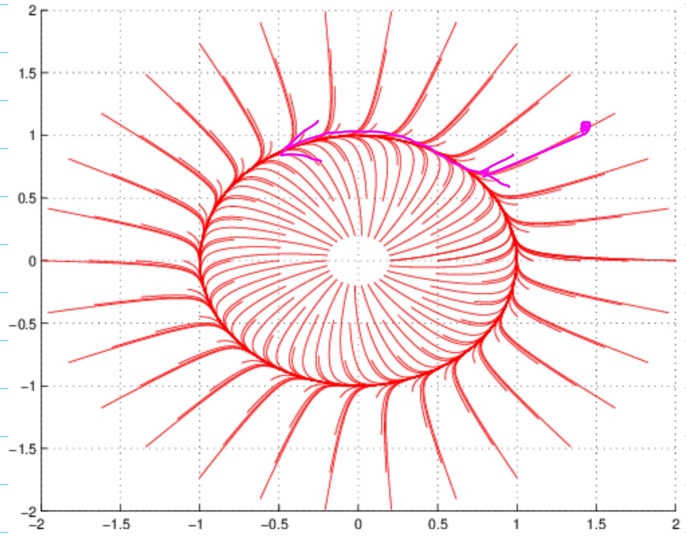
$$\underbrace{\dot{y} = f(y) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} y}_{\text{"rotation ODE":}} + \lambda(1-\|y\|_2^2)\,y \;, \quad D = \mathbb{R}^2$$

non-linear $= 0$, if $\|y(t)\|_2 = 1$

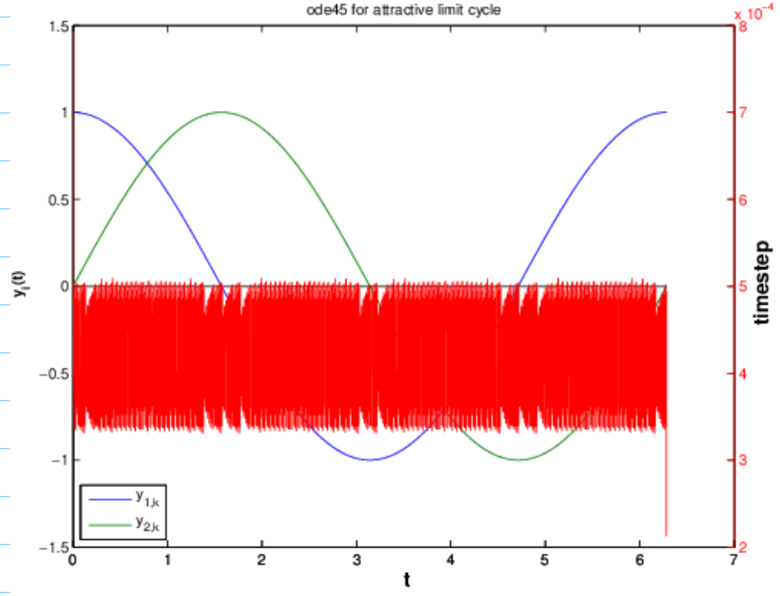
$$\|y(0)\|_2 = 1 \;\triangleright\; y(t) = \begin{bmatrix} \cos(t-\varphi) \\ \sin(t-\varphi) \end{bmatrix}, \quad \varphi \in \mathbb{R}$$



vectorfield $\mathbf{f}$ ($\lambda = 1$)


solution trajectories ($\lambda = 10$)

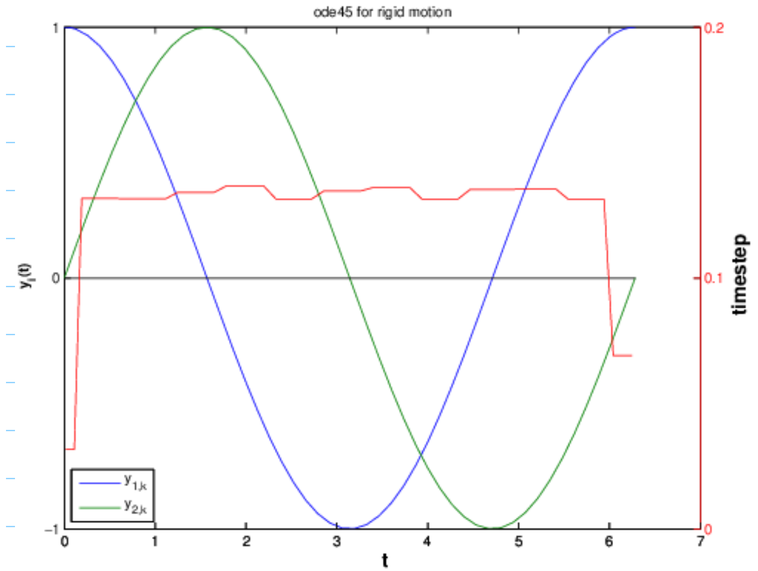
Ode45 adaptive explicit RK-SSM:



many (3794) steps ($\lambda = 1000$)

↓

stability-induced timestep constraint



accurate solution with few steps ($\lambda = 0$)

large timesteps

Notion 7.2.0.7. Stiff IVP

An initial value problem is called stiff, if stability imposes much tighter timestep constraints on *explicit single step methods* than the accuracy requirements.

③ **§ 7.2.0.8 Linearization of ODEs**

Autonomous ODE: $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, $\mathbf{f} : D \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$ smooth

$\mathbf{y}^* \in D$: $t \longmapsto \mathbf{y}(t)$ solution w/ $\mathbf{y}(0) = \mathbf{y}^*$

Linearization around $\mathbf{y}^*$:

$\mathbf{z}(t) := \mathbf{y}(t) - \mathbf{y}^*$: $\mathbf{z}(0) = 0$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{y}^* + \mathbf{z}) = \mathbf{f}(\mathbf{y}^*) + D\mathbf{f}(\mathbf{y}^*)\mathbf{z} + O(\|\mathbf{z}\|^2)$$

▶ The short-time evolution of $\mathbf{y}$ with $\mathbf{y}(0) = \mathbf{y}^*$ is approximately governed by the affine-linear ODE

$$\boxed{\dot{\mathbf{y}} = \mathbf{M}(\mathbf{y} - \mathbf{y}^*) + \mathbf{b}\,, \quad \mathbf{M} := D\mathbf{f}(\mathbf{y}^*) \in \mathbb{R}^{N,N}\,, \quad \mathbf{b} := \mathbf{f}(\mathbf{y}^*) \in \mathbb{R}^N}\,. \tag{7.2.0.10}$$

Ex 7.1.0.3 cnt'd: $f(y) = \lambda y(1-y)$, $\lambda \gg 1$

$y^* = 1$ $\quad [f(y^*) = 0]$

If $y(0) \approx y^*$: $\quad \dot{z} \approx -\lambda z$ [decay equ.]

§ 7.2.0.11 (Linearization of explicit RK-SSMs)

A commuting diagram: (timestep $h > 0$ fixed)

$$\begin{tikzcd}
\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) \arrow[r, "\text{linearize}", "\text{around } \mathbf{y}^*"'] \arrow[d, "\text{RK-SSM}"'] & \dot{\mathbf{z}} = D\mathbf{f}(\mathbf{y}^*)\mathbf{z} + \mathbf{f}(\mathbf{y}^*) \arrow[d, "\text{RK-SSM}"] \\
\text{Discrete evolution } \Psi^h \; [\Psi^h : D \to D] \arrow[r, "\text{linearize}", "\text{around } \mathbf{y}^*"'] & \tilde{\Psi}^h : D \to D
\end{tikzcd}$$

The discrete evolution of the RK-SSM for $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ in the state $\mathbf{y}^*$ is close to the discrete evolution of the same RK-SSM applied to the linearization (7.2.0.10) of the ODE in $\mathbf{y}^*$.

$$\left.\begin{aligned} \dot{\mathbf{z}} &= \mathbf{M}\mathbf{z} + \mathbf{b} \\ \mathbf{z}(0) &= \mathbf{z}_0 \end{aligned}\right\} \xrightarrow{\text{RK-SSM}} (\mathbf{z}_k)$$

$$\Rightarrow \quad \mathbf{w}_k = \mathbf{z}_k - \mathbf{z}_0 + \mathbf{M}^{-1}\mathbf{b}$$

$$\left.\begin{aligned} \dot{\mathbf{w}} &= \mathbf{M}\mathbf{w} \\ \mathbf{w}(0) &= \mathbf{M}^{-1}\mathbf{b} \end{aligned}\right\} \xrightarrow{\text{RK-SSM}} (\mathbf{w}_k)$$

The behavior of an explicit Runge-Kutta single-step method applied to $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ close to the state $\mathbf{y}^*$ is determined by the eigenvalues of the Jacobian $D\mathbf{f}(\mathbf{y}^*)$.

(4)

not a property of ODE

How to distinguish stiff initial value problems

An initial value problem for an autonomous ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ will probably be stiff, if, for substantial periods of time,

$$\min\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathbf{Df}(\mathbf{y}(t)))\} \ll 0, \tag{7.2.0.13}$$

and

$$\max\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathbf{Df}(\mathbf{y}(t)))\} \lesssim 0, \tag{7.2.0.14}$$

where $t \mapsto \mathbf{y}(t)$ is the solution trajectory and $\sigma(\mathbf{M})$ is the spectrum of the matrix $\mathbf{M}$, see .

set of eigenvalues

(13) $\exists$ negative eigenvalues λ w/ large modulus

→ "rapid decay property"

(14) no "physical blow-up" of solution

$h\lambda \in S_\Psi$: stability-induced timestep constraint for expl. RK-SSM

stability region

Ex. 7.2.0.15 (Predicting stiffness of non-linear IVPs)

- Ex. 7.0.0.1 : $N = 1$

IVP considered: $\dot{y} = f(y) := \lambda y^2(1-y)$, $\lambda := 500$, $y(0) = \frac{1}{100}$.

We find [$y^* = 1$]

$$f'(y) = \lambda(2y - 3y^2) \Rightarrow f'(1) = -\lambda . \ll 1$$

- Ex. 7.2.0.4 : $N = 2$

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y} + \lambda(1 - \|\mathbf{y}\|^2)\,\mathbf{y}, \quad \|\mathbf{y}_0\|_2 = 1 . \tag{7.2.0.5}$$

satisfies $\|\mathbf{y}(t)\|_2 = 1$ for all times.

$$\mathbf{Df}(\mathbf{y}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \lambda\left(-2\mathbf{y}\mathbf{y}^\top + (1 - \|\mathbf{y}\|_2^2\mathbf{I})\right).$$

$$\sigma(\mathbf{Df}(\mathbf{y})) = \{\underbrace{-\lambda - \sqrt{\lambda^2-1}}_{\ll 0}, \underbrace{-\lambda + \sqrt{\lambda^2-1}}_{\approx 0}\}, \text{ if } \|\mathbf{y}\|_2 = 1 .$$

$\lambda \gg 1 \Rightarrow$

↳ timestep constraint

Rem. 7.2.0.16 (Characterics of phenomena leading to stiff model IVPs)

Typical features of stiff IVPs:
- Presence of fast transients in the solution, see Ex. 7.1.0.3, Ex. 7.1.0.35,
- Occurrence of strongly attractive fixed points/limit cycles, see Ex. 7.2.0.4

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Please answer review questions

After you have watched the video take a short break and then try to answer the

review questions 7.2.0.17

Please do not flip pages of the lecture document, nor look at tablet notes.

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