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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 7.5: Splitting Methods

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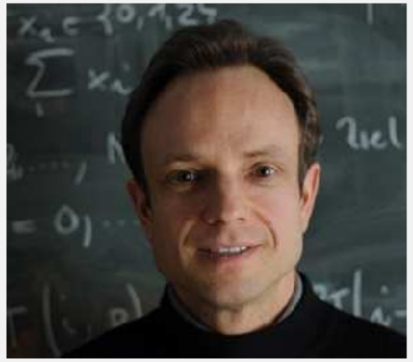

Date: April 7, 2021

Prerequisites.

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Dependencies. [Lecture → Section 6.3], [Lecture → Section 6.3.2]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

VII. Single-Step Methods for Stiff Initial-Value Problems

7.5. Splitting Methods

§ 7.5.0.1 (Splitting idea: Composition of partial evolutions)

ODE: $\dot{y} = f(y) + g(y)$, $f, g : D \subset \mathbb{R}^N \to \mathbb{R}^N$

Evolution operators

$\Phi_f : I \times D \to D \quad \longleftrightarrow \quad \dot{y} = f(y)$

$\Phi_g : I \times D \to D \quad \longleftrightarrow \quad \dot{y} = g(y)$

↳ Assumed to be available

Idea: Build single step methods (→ Def. 6.3.1.4) based on the following discrete evolutions

Lie-Trotter splitting: $\Psi^h = \Phi_g^h \circ \Phi_f^h$, (7.5.0.3)

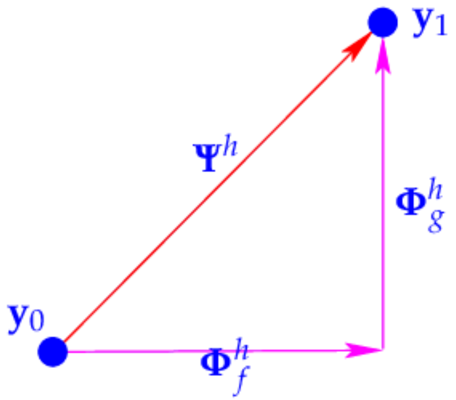
Strang splitting: $\Psi^h = \Phi_f^{h/2} \circ \Phi_g^h \circ \Phi_f^{h/2}$. (7.5.0.4)

↑ discrete evolutions

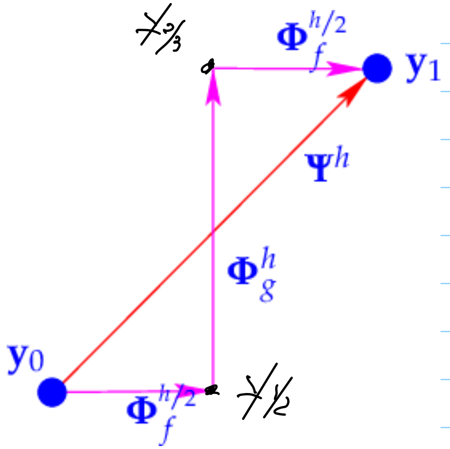
② ▷ Lie-Trotter splitting SSM, timestep $h>0$

$$y_0 \longrightarrow y_{1/2} := \Phi_f^h y_0 \longrightarrow y_1 = \Phi_g^h y_{1/2}$$
$$y_1 \longrightarrow y_{3/2} := \Phi_f^h y_1 \longrightarrow y_2 = \Phi_g^h y_{3/2}$$
$$\vdots$$

L.-T. (7.5.0.3) $\leftrightarrow$



Strang (7.5.0.4) $\leftrightarrow$



▷ Strang splitting SSM, timestep $h>0$

$$y_0 \to y_{1/3} := \Phi_f^{h/2} y_0 \to y_{2/3} = \Phi_g^h y_{1/3} \to y_1 := \Phi_f^{h/2} y_{2/3}$$
$$y_1 \to y_{4/3} := \Phi_f^{h/2} y_1 \longrightarrow y_{5/3} := \Phi_g^h y_{4/3} \to y_2 := \Phi_f^{h/2} y_{5/3}$$
$$y_2 \to y_{7/3} := \Phi_f^{h/2} y_2 \longrightarrow \dots$$

$$y_{4/3} = \Phi_f^h y_{3/3} \qquad y_{7/3} = \Phi_f^h y_{5/3}$$

Implementation:

$$y_0 \xrightarrow{\Phi_f^{h/2}} y_{1/3} \xrightarrow{\Phi_g^h} y_{2/3} \xrightarrow{\Phi_f^h} y_{4/3} \xrightarrow{\Phi_g^h} y_{5/3} \xrightarrow{\Phi_f^h} y_{7/3}$$
$$\cdots\cdots \xrightarrow{\Phi_f^h} y_\square \xrightarrow{\Phi_g^h} y_\square \xrightarrow{\Phi_f^{h/2}} y_M$$

▷ Differs from Lie-Trotter only in first & last step

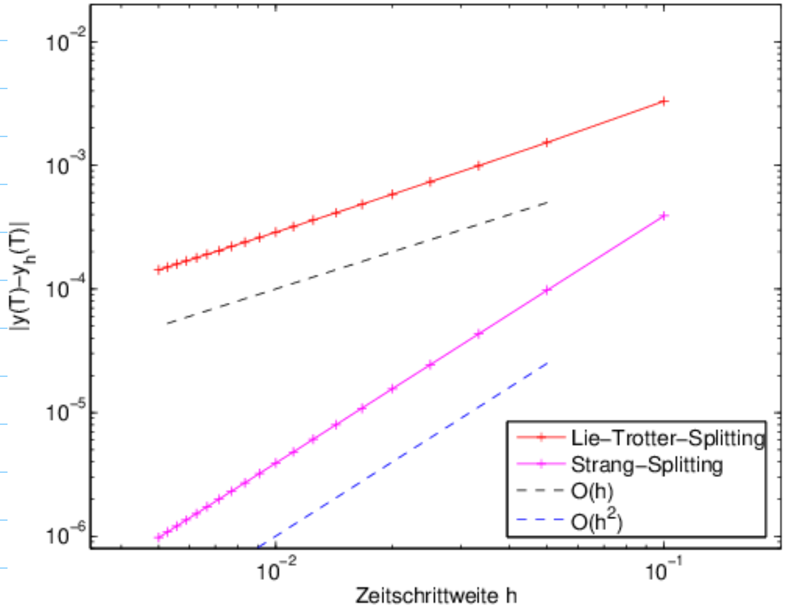
Exp. 7.5.0.5 (Convergence of simple splitting methods)

IVP: $\dot{y} = \underbrace{\lambda y(1-y)}_{=:f(y)} + \underbrace{\sqrt{1-y^2}}_{=:g(y)}$, $y(0)=0$.

▶ $\Phi_f^t y = \dfrac{1}{1+(y^{-1}-1)e^{-\lambda t}}$, $t>0, y\in]0,1]$ (logistic ODE (6.1.2.2))

▶ $\Phi_g^t y = \begin{cases} \sin(t+\arcsin(y)) & \text{, if } t+\arcsin(y) < \frac{\pi}{2}, \\ 1 & \text{, else,} \end{cases}$ $\quad t>0, y\in[0,1]$.

③



▷ L.-T. : order 1
Strang : order 2

Theorem 7.5.0.6. Order of simple splitting methods

Die single step methods defined by (7.5.0.3) *or* (7.5.0.4) *are of order* (→ *Def. 6.3.2.8*) 1 *and* 2, *respectively.*

↓ L.-T. ↓ Strang

§ 7.5.0.7 (Inexact splitting methods)

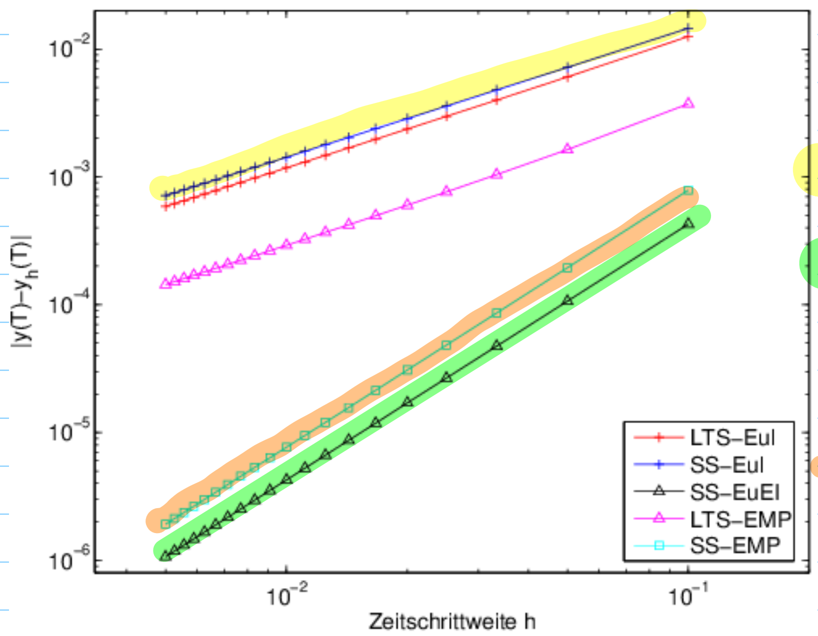
Idea: In (7.5.0.3)/(7.5.0.4) replace

exact evolutions ⟶ discrete evolutions

Φ_g^h, Φ_f^h ⟶ Ψ_g^h, Ψ_f^h .

Exp. 7.5.0.8 (Convergence of inexact splitting methods)

IVP : $\dot{y} = \underbrace{\lambda y(1-y)}_{=:f(y)} + \underbrace{\sqrt{1-y^2}}_{=:g(y)}$, $y(0) = 0$.



LTS-Eul	explicit Euler method (6.2.1.4) → $\Psi_{h,g}^h$, $\Psi_{h,f}^h$ + Lie-Trotter splitting (7.5.0.3)
SS-Eul	explicit Euler method (6.2.1.4) → $\Psi_{h,g}^h$, $\Psi_{h,f}^h$ + Strang splitting (7.5.0.4)
SS-EuEI	Strang splitting (7.5.0.4): explicit Euler method (6.2.1.4) ∘ exact evolution Φ_g^h ∘ implicit Euler method (6.2.2.2)
LTS-EMP	explicit midpoint method (6.2.3.3) → $\Psi_{h,g}^h$, $\Psi_{h,f}^h$ + Lie-Trotter splitting (7.5.0.3)
SS-EMP	explicit midpoint method (6.4.0.7) → $\Psi_{h,g}^h$, $\Psi_{h,f}^h$ + Strang splitting (7.5.0.4)

▷ Ψ_f, Ψ should match order of splitting scheme

Standard application scenario :

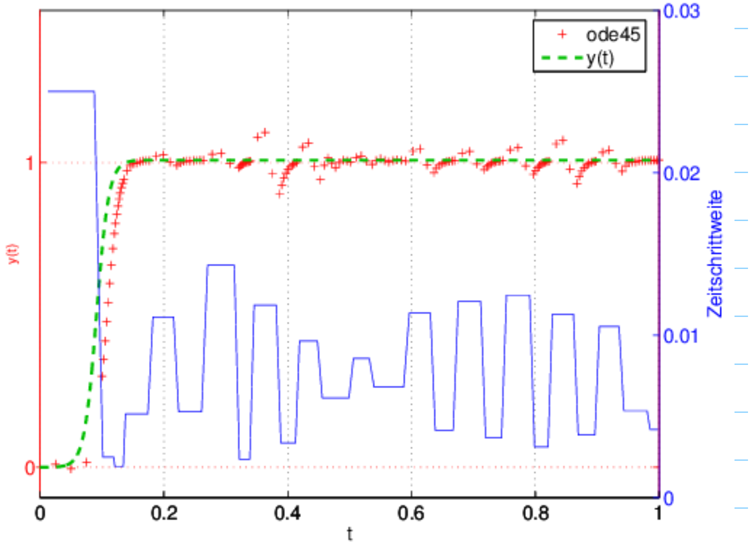
"Splittable" ODEs

$\dot{\mathbf{y}} = f(\mathbf{y}) + g(\mathbf{y})$ "'difficult" (e.g., stiff → Section 7.2) : $\dot{\mathbf{y}} = f(\mathbf{y})$ → stiff, but with an analytic solution; $\dot{\mathbf{y}} = g(\mathbf{y})$ "'easy", amenable to explicit integration.

Exp. 7.5.0.11 (Splitting off stiff components)

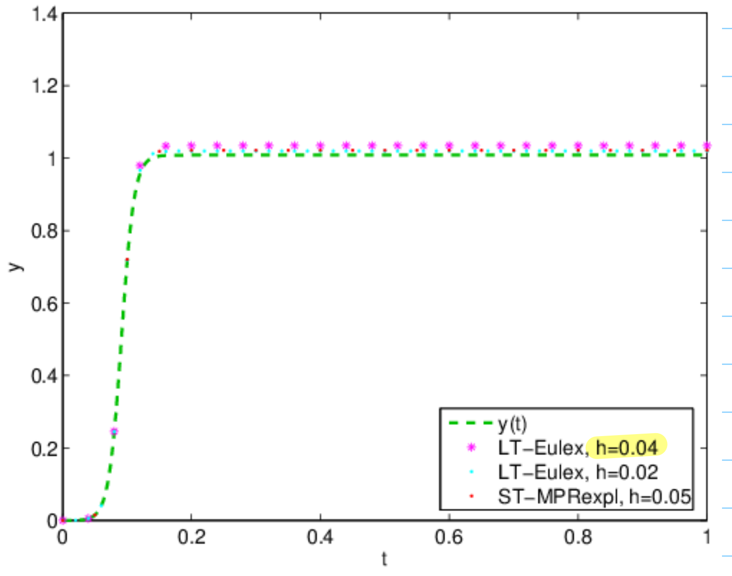
IVP: $\dot{y} = \underbrace{\lambda y(1-y)}_{f(y)} + \underbrace{\alpha \sin(y)}_{g(y)}$, $\lambda = 100$, $\alpha = 1$, $y(0) = 10^{-4}$.

small perturbation



Solution by **ode45**, see Ex. 7.0.0.1

↳ stiff



inexact splitting method: solution (y_k)

Ex. 7.5.0.12 (Splitting linear and decoupling terms)

ODE : $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) := -\mathbf{A}\mathbf{y} + \begin{bmatrix} g(y_1) \\ \vdots \\ g(y_N) \end{bmatrix}$, $\mathbf{A} = \mathbf{A}^\top \in \mathbb{R}^{N,N}$ positive definite (→) , $N \gg 1$ (7.5.0.13)

- $\lambda_{min}(A) \approx 1$, $\lambda_{max}(A) \approx N^2$
- $g : \mathbb{R} \to \mathbb{R}$ smooth , $|g'| \lessapprox 1$

$$D\mathbf{f}(\mathbf{y}) = -A + \begin{bmatrix} g'(y_1) & & \\ & \ddots & \\ & & g'(y_N) \end{bmatrix}$$

↓

spectrum $\approx \sigma(-A)$ $\Rightarrow$ stiff IVP

Splitting :

$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) := \underbrace{-\mathbf{A}\mathbf{y}}_{f_1(y)} + \underbrace{\begin{bmatrix} g(y_1) \\ \vdots \\ g(y_N) \end{bmatrix}}_{f_2(y)}$, $\mathbf{A} = \mathbf{A}^\top \in \mathbb{R}^{N,N}$ positive definite (→) , (7.5.0.13)

$\dot{y} = f_1(y)$: stiff
use L-stable IRK-SSM

$\dot{y} = f_2(y)$ non-stiff
use explicit RK-SSM

5 Concrete: Strang splitting & order-2 methods

Please answer review questions

After you have watched the video take a short break and then try to answer the

review questions 7.5.0.14

Please do not flip pages of the lecture document, nor look at tablet notes.