ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

# Course Video
# Section 9.2.1: Heat Equation

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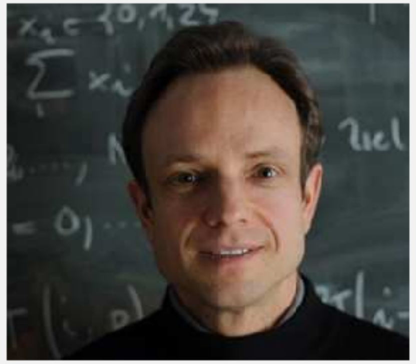


Date: April 28, 2021



**Prerequisites.**

- Differential operators **grad**, div and differentiation in time.

**Dependency.** [Lecture → Section 1.6] and [Lecture → Section 1.7]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

Duration : 15 minutes



# VI. Second-Order Linear Evolution Problems

time-dependent / transient

## 9.1. Time-Dependent Boundary Value Problems

▷ posed on space-time cylinder

Domain $\tilde{\Omega} = \Omega \times ]0,T[$

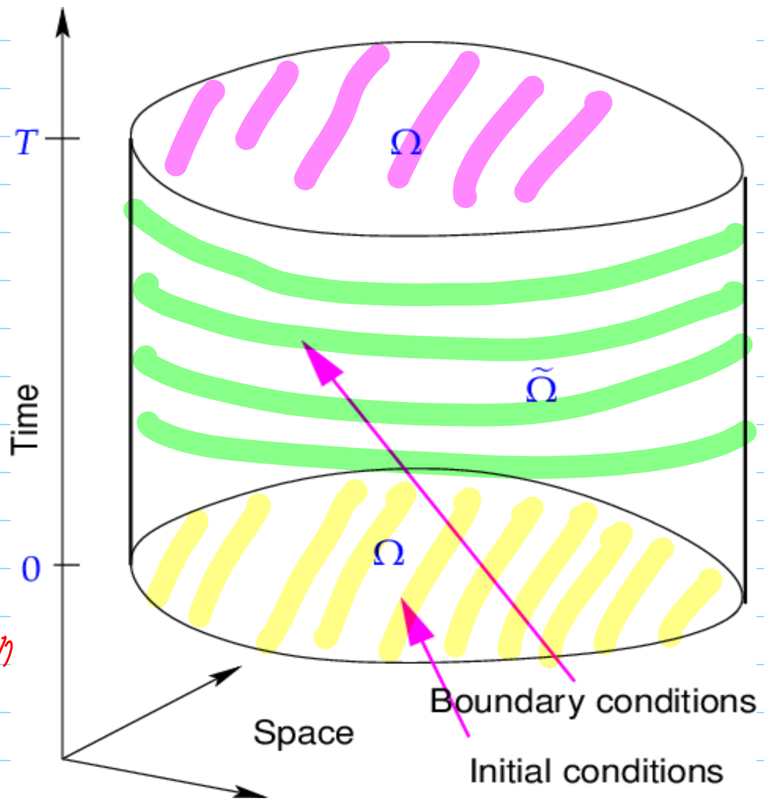
$\Omega \;\hat{=}$ spatial domain

$T \;\hat{=}$ final time, $T > 0$

solution: $u = u(x,t)$

$x \in \Omega,\quad 0 \leq t \leq T$

- $\Omega \times \{0\}$ → initial conditions
- $\partial\Omega \times ]0,T[$ → (spatial) boundary condition
- NO "final conditions"



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▷ Special about time ⇒ has a direction [causality]

# 9.2. Parabolic Initial-Boundary Value Problems

## 9.2.1. Heat Equation

Balance law:

*Conservation of energy:* independent of time

$$\frac{d}{dt}\int_V \rho u \,\mathrm{d}x + \int_{\partial V} \mathbf{j}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_V f\,\mathrm{d}x \quad \text{for all "control volumes" } V\,. \tag{9.2.1.3}$$

energy stored in $V$ — power flux through $\partial V$ — heat generation in $V$

$\rho = \rho(\mathbf{x})$: (spatially varying) heat capacity ($[\rho] = \mathrm{JK}^{-1}$), uniformly positive, *cf.* (1.6.0.6).

$$\triangleright \quad \int_V \frac{\partial}{\partial t}(\rho u)(x,t)\,dx + \underbrace{\int_{\partial V} j\cdot n\,dS}_{=\int_V \operatorname{div} j\,dx} = \int_V f\,dx \quad \forall V$$

Localization ⇓

$$\frac{\partial}{\partial t}(\rho u)(\boldsymbol{x},t) + (\operatorname{div}_{\boldsymbol{x}}\mathbf{j})(\boldsymbol{x},t) = f(\boldsymbol{x},t) \quad \text{in } \tilde{\Omega}\,. \tag{9.2.1.5}$$

③ + Fourier law $j = -\kappa(x)\,\mathrm{grad}\,u$

$$\boxed{\frac{\partial}{\partial t}(\rho u) - \mathrm{div}(\kappa(x)\,\mathbf{grad}\,u) = f \quad \text{in} \quad \tilde{\Omega} := \Omega\times]0,T[} \,. \tag{9.2.1.6}$$

• Spatial b.c. on $\partial\Omega \times ]0,T[$

For second order parabolic evolutions we can/must use the *same* spatial boundary conditions as for stationary second order elliptic boundary value problems.

On $\partial\Omega]0,T[$ we can impose any of the boundary conditions discussed in Section 1.7:
- Dirichlet boundary conditions $u(x,t) = g(x,t)$, see (7.1.1.7) (fixed surface temperature),
- Neumann boundary conditions $\mathbf{j}(x,t)\cdot n = -h(x,t)$ (fixed heat flux through surface),
- radiation boundary conditions $\mathbf{j}(x,t)\cdot n = \Psi(u(x,t))$,

and any combination of these as discussed in Ex. 1.7.0.10, yet, *only one* of them at any part of $\partial\Omega\times]0,T[$, see Rem. 1.7.0.9.

same as in stationary setting

• Initial conditions: ("temporal b.c.")

$$\underbrace{u(x,0) = u_0(x)}_{\text{IBVP}}$$

e.g.

$$\frac{\partial}{\partial t}(\rho(x)u) - \mathrm{div}(\kappa(x)\,\mathbf{grad}\,u) = f \quad \text{in} \quad \tilde{\Omega} := \Omega\times]0,T[\,, \tag{9.2.1.6}$$

$$u(x,t) = g(x,t) \quad \text{for} \quad (x,t)\in\partial\Omega\times]0,T[\,, \tag{9.2.1.7}$$

$$u(x,0) = u_0(x) \quad \text{for all} \quad x\in\Omega\,. \tag{9.2.1.8}$$

Comp. cond.:

$u_0(x) = g(x,0)$, $x \in \partial\Omega$

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# Review questions 9.1.0.7 & 9.2.1.14

[ Answer without recourse to any notes ]

A:

Draw a space-time cylinder and mark where initial conditions ~~are~~ and (spatial) boundary conditions are imposed.

B:

Evolution problems can also be posed on time-dependent spatial domains. For $d = 2$ sketch a generalization of the space-time cylinder for that case.
Outline an approach that can convert an evolution problem posed on a time-dependent spatial domain into one living on a space-time cylinder.

C:

You can think of a reversible physical systems as those for which you could not tell whether a video recording of them is played forward or backward.
Give examples of reversible and irreversible physical systems in our everyday world.

D

Cast the statement

The change is thermal energy stored in a body is balanced by the heat flow through its surface

into a mathematical formula and show that it holds for the linear heat equation with pure Neumann spatial boundary conditions.

E:

In a model for linear transient heat conduction in a homogeneous body let $u = u(\boldsymbol{x},t) : \Omega \times [0,T] \to \mathbb{R}$ designated the temperature distribution. What is the physical meaning of the following spatial boundary conditions, here written in non-dimensional form:

1. $u(\boldsymbol{x},t) = g(\boldsymbol{x},t)$, $(\boldsymbol{x},y) \in \partial\Omega \times [0,T]$ (Dirichlet b.c.),
2. $\mathbf{grad}\, u(\boldsymbol{x},t) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0$, $(\boldsymbol{x},y) \in \partial\Omega \times [0,T]$, (homogeneous Neumann b.c.),
3. $\mathbf{grad}\, u(\boldsymbol{x},t) \cdot \boldsymbol{n}(\boldsymbol{x}) = u(\boldsymbol{x},t)$, $(\boldsymbol{x},y) \in \partial\Omega \times [0,T]$, (impedance b.c.)?