ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 9.2.3: Stability of Parabolic Evolution Problems

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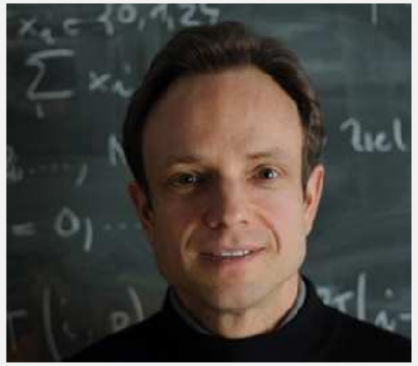

Prerequisites.

- (Bi-)linear forms
- Calculus in 1D

Dependency. [Lecture → Section 9.2.2]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

Duration: 20 minutes

VI. Second-Order Linear Evolution Problems

9.2. Parabolic Initial Boundary Value Problems

9.2.3. Stability of Parabolic Evolution Problems

↳ $\|u(t)\|$ controlled by data $\ell(t)$, u_0

Also: structural properties of u

s.p.d. bilinear forms

$$t\in]0,T[\mapsto u(t)\in V_0 \;:\; \begin{cases}\dfrac{d}{dt}\mathsf{m}(u(t),v)+\mathsf{a}(u(t),v)=\cancel{\ell(t)(v)}\;0 & \forall v\in V_0,\\ u(0)=u_0\in V_0 .\end{cases} \qquad (9.2.2.9)$$

Heat equ. context: $V_0 = H_0^1(\Omega)$, $\|\cdot\|_m \leftrightarrow L^2$-norm, $\|\cdot\|_a \sim H^1$-seminorm

P.F. inequality: $\|v\|_{L^2(\Omega)} \leq \operatorname{diam}(\Omega)\,|v|_{H^1} \quad \forall v\in H_0^1(\Omega)$

$\Updownarrow$ $\|\cdot\|_m$ $\qquad\qquad$ $\Updownarrow$ $\|\cdot\|_a$

(2)

Assume: $\exists\, \gamma > 0: \quad \gamma \|v\|_m^2 \leq \|v\|_a^2 \quad \forall v \in V_0$

Trick: $\quad w(t) = e^{\gamma t} u(t) \in V_0$

Product rule: $\quad \dot{w}(t) = e^{\gamma t}(\dot{u} + \gamma u) = e^{\gamma t}\dot{u} + \gamma w$

$$m(\dot{w}, v) = e^{\gamma t} m(\dot{u}, v) + \gamma m(w, v)$$

$$(6.2.2.9) \triangleright = e^{\gamma t}(-a(u,v)) + \gamma m(w,v) \qquad (*)$$

$$= -a(w,v) + \gamma m(w,v)$$

$$\frac{d}{dt} \tfrac{1}{2}\|w\|_m^2 = \frac{d}{dt} \tfrac{1}{2} m(w,w)$$

$$= \cancel{\tfrac{1}{2}\cdot 2}\, m(\dot{w}, w)$$

Product rule for bilinear form

$$= -\|w\|_a^2 + \gamma \|w\|_m^2 \leq 0$$

$$\triangleright \quad \|w(t)\|_m \leq \|w(0)\|$$

$$\|u(t)\|_m \leq e^{-\gamma t}\|u_0\|_m$$

$$\frac{d}{dt} \tfrac{1}{2}\left(\|w\|_a^2 - \gamma \|w\|_m^2\right) = -\frac{d}{dt} \tfrac{1}{2} m(\dot{w}, w)$$

$$= -\|\dot{w}\|_m^2 \leq 0$$

$$\triangleright \quad \|w(t)\|_a^2 - \gamma \|w(t)\|_m^2 \leq \|w(0)\|_a^2 - \gamma \|w(0)\|_m^2$$

$$\triangleright \quad \|w(t)\|_a^2 \leq \|w(0)\|_a^2 - \gamma \underbrace{\left(\|w(0)\|_m^2 - \|w(t)\|_m^2\right)}_{\geq 0}$$

Lemma 9.2.3.8. Decay of solutions of parabolic evolutions

For $f \equiv 0$ the solution $u(t)$ of (7.1.2.4) satisfies

$$\|u(t)\|_m \leq e^{-\gamma t}\|u_0\|_m \quad , \quad \|u(t)\|_a \leq e^{-\gamma t}\|u_0\|_a \quad \forall t \in]0, T[\,,$$

where $\gamma > 0$ is the constant from (7.1.3.1), and $\|\cdot\|_a$, $\|\cdot\|_m$ stand for the energy norms induced by $\mathsf{a}(\cdot,\cdot)$ and $\mathsf{m}(\cdot,\cdot)$, respectively.

Dissipation of energy in parabolic evolutions

▶ Exponential decay of energy during parabolic evolution without excitation ("Parabolic evolutions dissipate energy")

Supplement [Q&A, April 27, 2020]

$(*) \rightarrow$ Also w solves an abstract parabolic evolution problem

$$m(\dot{w}, v) + \tilde{a}(w,v) = 0 \;, \quad \tilde{a} = a - \gamma m \;\hat{=}\; \text{positive semidef}$$

$$\frac{d}{dt} \tfrac{1}{2} \tilde{a}(w,w) = \tilde{a}(\dot{w}, w) \overset{v = \dot{w}}{=} -m(\dot{w}, \dot{w}) = -\|\dot{w}\|_m^2 \leq 0$$

Review questions 9.2.3.13

[To be answered without aids]

A :

Compute the solution of the 1D parabolic evolution problem

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{in }]0,1[\times[0,T]\,, \\ u(0,t) = u(1,t) &= 0 \quad \text{for all} \quad 0<t<T\,, \\ u(x,0) &= \{x \mapsto \sin(\pi x)\} \quad \text{for all} \quad 0<x<1\,. \end{aligned}$$

Hint. Use the separation of variables approach with $u(x,t) = \sin(\pi x)\psi(t)$ and determine the function ψ.

B :

For a sufficiently smooth function $u = u(\boldsymbol{x},t)$ compute the partial derivative $\frac{\partial w}{\partial t}(\boldsymbol{x},t)$ for $w(\boldsymbol{x},t) := \exp(\gamma t)u(\boldsymbol{x},t)$ in terms of partial derivatives of u.

C :

For a domain $\Omega \subset \mathbb{R}^2$ and a vector $\boldsymbol{c} \in \mathbb{R}^2$ we consider the bilinear form

$$\mathrm{b}(u,v) := \int_\Omega \mathbf{grad}\, u(\boldsymbol{x}) \cdot \boldsymbol{c}\, v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\,, \quad u,v \in H^1(\Omega)\,.$$

Express $\frac{d}{dt}b(u,u)$ for a sufficiently smooth function $u : \Omega \times [0,T] \to \mathbb{R}$ using partial derivatives of u.

D :

For a bounded domain $\Omega \subset \mathbb{R}^2$ consider the parabolic evolution problem

$$\begin{aligned} \frac{\partial}{\partial t}(\rho(\boldsymbol{x})u) - \Delta u &= 0 \quad \text{in} \quad \widetilde{\Omega} := \Omega\times]0,T[\,, \\ \mathbf{grad}\, u(\boldsymbol{x},t)\cdot \boldsymbol{n}(\boldsymbol{x}) &= 0 \quad \text{for} \quad (\boldsymbol{x},t) \in \partial\Omega\times]0,T[\,, \\ u(\boldsymbol{x},0) &= u_0(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x}\in\Omega\,, \end{aligned}$$

with (uniformly positive) coefficient function $\rho : \Omega \to \mathbb{R}$ and given initial data $u_0 \in H^1(\Omega)$. Can we expect $u(\boldsymbol{x},T) \to 0$ for $T \to \infty$? What kind of exponential decay can we observe in this case?