ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 9.2.4: Spatial Semi-Discretization: Method of Lines

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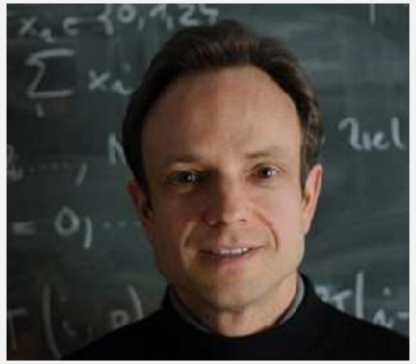

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Prerequisites.

- Basic knowledge about ordinary differential equations

Dependency. [Lecture → Section 9.2.2], [Lecture → Section 2.2], and [Lecture → Section 2.6]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

Duration: 13 minutes

VI. Second-Order Linear Evolution Problems

9.2. Parabolic Initial Boundary Value Problems

9.2.4. Spatial Semi-Discretization: Method of Lines

→ Spatial Galerkin discretization* of

$$t \in]0,T[\mapsto u(t) \in V_0 \ : \quad \begin{cases} \mathsf{m}(\dot{u}(t),v) + \mathsf{a}(u(t),v) = \ell(t)(v) & \forall v \in V_0\,, \\ u(0) = u_0 \in V_0\,. \end{cases} \qquad (9.2.29)$$

Step I: Replace $V_0 \rightarrow V_{0,h} \subset V_0$, $\dim V_0 =: N < \infty$
↳ independent of time

1st step: replace V_0 with a finite dimensional subspace $V_{0,h}$, $N := \dim V_{0,h} < \infty$

▶ (Spatially) discrete parabolic evolution problem

$$t \in]0,T[\mapsto u_h(t) \in V_{0,h} \ : \quad \begin{cases} \mathsf{m}(\dot{u}_h(t),v_h) + \mathsf{a}(u_h(t),v_h) = \ell(t)(v_h) & \forall v_h \in V_{0,h}\,, \\ u_h(0) = \text{projection/interpolant of } u_0 \text{ in } V_{0,h}\,. \end{cases} \qquad (9.2.4.1)$$

$\rightarrow$ $u_h(t) \hat{=}$ a function-space valued function of time

2nd step: ordered basis $\mathfrak{B} = \{b_h^1, \dots, b_h^N\} \subset V_{0,h}$ [independent of time]

$$u_h(t) = \sum_{i=1}^{N} \mu_i(t)\, b_h^i$$

↳ t-dependent coefficients

Use bi-linearity of m & a:

$$\sum_{i=1}^{N} \dot{\mu}_i(t)\, m(b_h^i, b_h^j) + \sum \mu_i(t)\, a(b_h^i, b_h^j) = \ell(t)(b_h^j) \quad \forall v_h \in V_{0,h}$$

$j = 1, \dots, N$

Method-of-lines ordinary differential equation

Combining (7.1.4.1) and (7.1.4.2) we obtain (MOL-ODE)

$$(7.1.4.1) \quad \Rightarrow \quad \begin{cases} \mathbf{M}\left\{\frac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) & \text{for } 0 < t < T\,, \\ \vec{\mu}(0) = \vec{\mu}_0\,. \end{cases} \qquad (7.2.4.4)$$

with

- ▷ s.p.d. stiffness matrix $\mathbf{A} \in \mathbb{R}^{N,N}$, $(\mathbf{A})_{ij} := \mathsf{a}(b_h^j, b_h^i)$ (independent of time),
- ▷ s.p.d. mass matrix $\mathbf{M} \in \mathbb{R}^{N,N}$, $(\mathbf{M})_{ij} := \mathsf{m}(b_h^j, b_h^i)$ (independent of time),
- ▷ source (load) vector $\vec{\varphi}(t) \in \mathbb{R}^N$, $(\vec{\varphi}(t))_i := \ell(t)(b_h^i)$ (time-dependent),
- ▷ $\vec{\mu}_0 \hat{=}$ coefficient vector of a projection of u_0 onto $V_{0,h}$.

Note:

$\boxed{\text{(7.1.4.4) is an ordinary differential equation (ODE) for } t \mapsto \vec{\mu}(t) \in \mathbb{R}^N}$

Relate to "std. ODE": $\dot{y} = f(t, y)$

$$\dot{\vec{\mu}} = M^{-1}(\vec{\varphi}(t) - A\vec{\mu}(t))$$

$\vec{\mu}(t)$ a function of $\underbrace{\text{continuous time}}$

MOL-ODE is semi-discrete

③ Review questions 9.2.4.7

A :

The method-of-lines spatial Galerkin discretization of the linear evolution problem in variational form

$$t \in]0,T[\mapsto u(t) \in V_0: \quad \mathsf{m}(\dot{u}(t),v) + \mathsf{a}(u(t),v) = \ell(t)(v) \quad \forall v \in V_0 ,$$

leads to the ordinary differential equation

$$\mathbf{M}\frac{d\vec{\mu}}{dt}(t) + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t)$$

for the basis expansion coefficient vector $\vec{\mu} \in \mathbb{R}^N$ of the time-dependent semi-discrete Galerkin solution $u_h \in V_{0,h}$.
Give formulas for the entries of the matrices $\mathbf{M}$, $\mathbf{A}$ and of the vector $\vec{\varphi}$.

B :

Verify the following result by direct computation.

Theorem 6.2.4.8. Variation-of-constants formula

For a fixed matrix $\mathbf{A} \in \mathbb{R}^{N,N}$, $N \in \mathbb{N}$, *and a Lipschitz continuous function* $\mathbf{g} : [0,\infty[\to \mathbb{R}^N$, *the initial-value problem*

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{g}(t) \quad , \quad \mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^N ,$$

has the unique solution

$$\mathbf{y}(t) = \exp(\mathbf{A}t)\left(\mathbf{y}_0 + \int_0^t \exp(-\mathbf{A}\tau)\,\mathbf{g}(\tau)\,d\tau\right) \quad t \geq 0 .$$

In this theorem $\exp(\cdot)$ denotes the matrix exponential defined by the exponential series in the very same way as e^z for any $z \in \mathbb{C}$.

C :

Use Thm. 6.2.4.8 to write down the solution of the method-of-lines initial-value problem

$$\begin{cases} \mathbf{M}\left\{\frac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) & \text{for } 0 < t < T , \\ \vec{\mu}(0) = \vec{\mu}_0 . \end{cases}$$

D :

Consider

$$\begin{cases} \mathbf{M}\left\{\frac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) & \text{for } 0 < t < T , \\ \vec{\mu}(0) = \vec{\mu}_0 , \end{cases} \qquad (6.2.4.4)$$

in the special case $\vec{\varphi} \equiv 0$. We replace the initial value $\vec{\mu}_0 \in \mathbb{R}^N$ with $\tilde{\vec{\mu}}_0 \in \mathbb{R}^N$ and, thus, obtain the perturbed solution $t \mapsto \tilde{\vec{\mu}}(t)$. Estimate $(\vec{\mu}(t) - \tilde{\vec{\mu}}(t))^\top \mathbf{A}(\vec{\mu}(t) - \tilde{\vec{\mu}}(t))$.

E :

The spatial Galerkin semi-discretization of the evolution problem

$$t \in]0,T[\mapsto u(t) \in V_0 \quad : \quad \begin{cases} \frac{d}{dt}\mathsf{m}(u(t),v) + \mathsf{a}(u(t),v) = \ell(t)(v) & \forall v \in V_0 , \\ u(0) = u_0 \in V_0 . \end{cases}$$

leads to an ordinary differential equation, which can be written in the form $\frac{d}{dt}\vec{\mu} = F(\vec{\mu})$. Give an expression for F with detailed formulas for all terms.