ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 9.2.5: Ordinary Differential Equations

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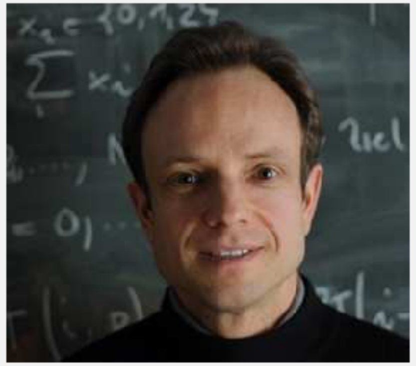


Prerequisites.

- Linear algebra: diagonalization of matrices

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 $\rightarrow$ New Chapter 9 , Old Chapter 8 $\rightarrow$ New Chapter 11
Trailing digits in labels are not affected.

Duration: 20 minutes

VI. Second-Order Linear Evolution Problems

9.2. Parabolic Initial Boundary Value Problems

9.2.5. Ordinary Differential Equations [Supplement Section]

ODE: $\dot{\mathbf{u}}(t) = \mathbf{f}(t, \mathbf{u}(t))$

$\mathbf{u}: I \subset \mathbb{R} \to \mathbb{R}^N$ $\quad$ $\mathbf{f}: I \times \mathbb{R}^N \to \mathbb{R}^N$

state space $\quad$ right-hand-side vectorfield (assumed to be Lipschitz cont.)

(2)

Method-of-lines ordinary differential equation

Combining (7.1.4.1) and (7.1.4.2) we obtain

$$\text{(7.1.4.1)} \quad \Rightarrow \quad \begin{cases} \mathbf{M}\left\{\frac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) & \text{for } 0 < t < T\,, \\ \vec{\mu}(0) = \vec{\mu}_0\,. \end{cases} \qquad (7.2.4.4)$$

with

- ▷ s.p.d. stiffness matrix $\mathbf{A} \in \mathbb{R}^{N,N}$, $(\mathbf{A})_{ij} := \mathsf{a}(b_h^j, b_h^i)$ (independent of time),
- ▷ s.p.d. mass matrix $\mathbf{M} \in \mathbb{R}^{N,N}$, $(\mathbf{M})_{ij} := \mathsf{m}(b_h^j, b_h^i)$ (independent of time),
- ▷ source (load) vector $\vec{\varphi}(t) \in \mathbb{R}^N$, $(\vec{\varphi}(t))_i := \ell(t)(b_h^i)$ (time-dependent),
- ▷ $\vec{\mu}_0 \triangleq$ coefficient vector of a projection of u_0 onto $V_{0,h}$.

Note:

(7.1.4.4) is an ordinary differential equation (ODE) for $t \mapsto \vec{\mu}(t) \in \mathbb{R}^N$

▷ State space $\mathbb{R}^N$, $u(t) \longleftrightarrow \vec{\mu}(t)$

$f(t,u) \longleftrightarrow M^{-1}(\vec{\varphi} - A\vec{\mu}(t))$

ODE has many solutions: $\dot{u} = \lambda u$ ▷ $u(t) = Ce^{\lambda t}$ $\forall \lambda \in \mathbb{R}$

Unique solutions for initial value problem:

$\dot{u} = f(t,u)$ + $u(t_0) = u_0 \in \mathbb{R}^N$, $t_0 \in I$ (IVP)

▷ Defines evolution operator $\phi : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$

$\phi^{t_0,t} u_0 := u(t)$ where u solves IVP

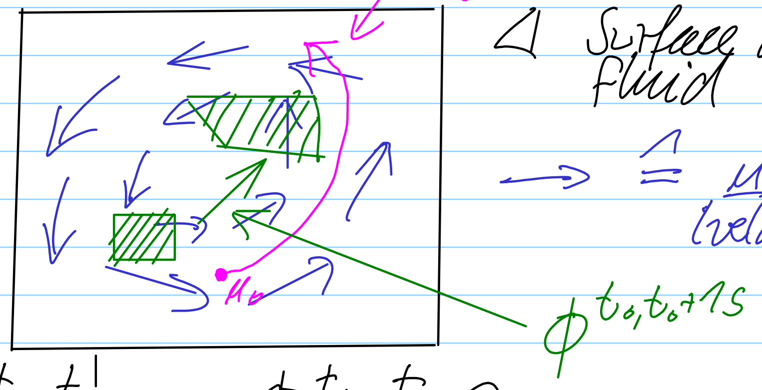
- For fixed t_0, u_0: $\underbrace{t \to \phi^{t_0,t} u_0}$ $\triangleq$ solution of (IVP)

trajectory: solution curve for ODE

- For fixed t_0, t: $\underbrace{u_0 \to \phi^{t_0,t} u_0 \in \mathbb{R}^N}$

bijective mapping $\mathbb{R}^N \to \mathbb{R}^N$ (deformation)

"Mental image" $N = 2$



◁ Surface of a moving fluid

$\rightarrow \;\triangleq\; u \to f(u)$ (velocity field)

- $\phi^{t',t} \circ \phi^{t_0,t'} = \phi^{t_0,t}$
- $\phi^{t,t} = Id$

$\Rightarrow (\phi^{t_0,t})^{-1} = \phi^{t,t_0}$

ODE from ϕ:

$f(t, \phi(t,t_0,u_0)) = \frac{\partial \phi}{\partial t}(t_0,t,u_0)$

ODE $\dot{u} = f(t,u)$ $\Longleftrightarrow$ Evol.-Op. ϕ

③

Definition 6.1.1.9. Linear ODE

An ODE $\dot{\mathbf{u}} = \mathbf{f}(t, \mathbf{u})$ as introduced in § 7.1.5.1 is linear, if $\mathbf{f}(t, \mathbf{u}) = \mathbf{A}(t)\mathbf{u}$ with a continuous function $\mathbf{A} : I \to \mathbb{R}^{N,N}$.

If $A(t) = A$ [autonomous linear ODE]

→ Solve by diagonalization:

$$\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}, \quad \mathbf{S} \in \mathbb{R}^{N,N} \text{ regular}, \quad \mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^{N,N}, \tag{9.2.5.13}$$

→ Decoupled scalar ODEs for components of $w(t) = S^{-1}u(t)$

$$\dot{w} = Dw$$

$$\mathbf{u}(t) = \mathbf{S} \begin{bmatrix} e^{\lambda_1 (t-t_0)} & & \\ & \ddots & \\ & & e^{\lambda_N (t-t_0)} \end{bmatrix} \mathbf{S}^{-1} \mathbf{u}_0 \,. \tag{9.2.5.14}$$

$\uparrow$ initial value

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Review questions 9.2.5.15

[Answer without looking at lecture documents]

A:

Let $\Omega \subset \mathbb{R}^3$ be the volume of space occupied by a container filled with a fluid. Let $v : \overline{\Omega} \to \mathbb{R}^3$ be a stationary velocity field describing the movement of the fluid.

- Which property of v ensures that the fluid stays in the container?
- What is the physical interpretation of the solutions of the ODE $\dot{\mathbf{y}} = v(\mathbf{y})$?

B:

Determine the evolution operator $\mathbf{\Phi} : \mathbb{R} \times \mathbb{R} \times D$ for the scalar ODE $\dot{y} = \cos^2 y$ on the state space $D :=]-\pi/2, \pi/2[$.

Hint. $\tan' = \cos^{-2}$.

C:

The evolution of operator for an ODE on the state space $\mathbb{R}^2$ is given by

$$\mathbf{\Phi}(s,t,\mathbf{y}) = \begin{bmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{bmatrix} \mathbf{y}\,, \quad s,t6 \in \mathbb{R}\,, \quad \mathbf{y} \in \mathbb{R}^2\,.$$

- Verify the so-called flow properties $\mathbf{\Phi}(t,t,\mathbf{y}) = \mathbf{y}$ and $\mathbf{\Phi}(s,t,\mathbf{\Phi}(r,s,\mathbf{y})) = \mathbf{\Phi}(r,t,\mathbf{y})$ for all $s,r,t \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^2$.
- Find the associated ODE.