ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 9.2.7: Timestepping for Method-of-Lines ODE

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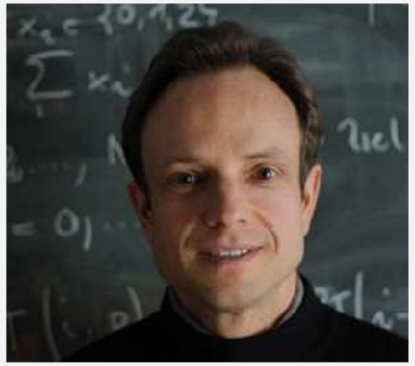

Prerequisites.

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Dependency. [Lecture → Section 9.2.2] and [Lecture → Section 9.2.6]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:

Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11

Trailing digits in labels are not affected.

Duration: 57 minutes

VI. Second-Order Linear Evolution Problems

9.2. Parabolic Initial Boundary Value Problems

9.2.7. Timestepping for Method-of-Lines ODE

Method-of-lines ordinary differential equation

Combining (7.1.4.1) and (7.1.4.2) we obtain [A, M s.p.d.]

$$(9.2.4.1) \quad \Rightarrow \quad \begin{cases} \mathbf{M}\left\{\frac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) & \text{for } 0 < t < T\,, \\ \vec{\mu}(0) = \vec{\mu}_0\,. \end{cases} \qquad (9.2.4.4)$$

with

▷ s.p.d. stiffness matrix $\mathbf{A} \in \mathbb{R}^{N,N}$, $(\mathbf{A})_{ij} := \mathsf{a}(b_h^j, b_h^i)$ (independent of time),

▷ s.p.d. mass matrix $\mathbf{M} \in \mathbb{R}^{N,N}$, $(\mathbf{M})_{ij} := \mathsf{m}(b_h^j, b_h^i)$ (independent of time),

▷ source (load) vector $\vec{\varphi}(t) \in \mathbb{R}^N$, $(\vec{\varphi}(t))_i := \ell(t)(b_h^i)$ (time-dependent),

▷ $\vec{\mu}_0 \mathrel{\hat{=}}$ coefficient vector of a projection of u_0 onto $V_{0,h}$.

Note:

(7.1.4.4) is an ordinary differential equation (ODE) for $t \mapsto \vec{\mu}(t) \in \mathbb{R}^N$

(2)

9.2.7.1. Single-Step-Methods (SSM) Applied to Method-of-Lines ODE

$$\mathbf{M}\left\{\frac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \quad \text{for } 0 < t < T\,, \qquad (9.2.4.4)$$
$$\vec{\mu}(0) = \vec{\mu}_0\,.$$

Std. ODE : $\dot{u} = f(t,u)$

$$\dot{\vec{\mu}}(t) = M^{-1}(\vec{\varphi} - A\vec{\mu}(t))$$

(i) Explicit Euler : $[\Psi^{t,t+\tau}u = u + \tau f(t,u)]$

$\triangleright$ $\vec{\mu}^{(j)} = \Psi^{t_{j-1},t_j}\vec{\mu}^{(j-1)} = \vec{\mu}^{(j-1)} + \tau M^{-1}(\vec{\varphi}(t_{j-1}) + A\vec{\mu}^{(j-1)})$

$\tau := t_j - t_{j-1}$

(ii) Implicit Euler : $[\Psi^{t,t+\tau}u = u + \tau f(t, \Psi^{t,t+\tau}u)]$

$\triangleright$ $\vec{\mu}^{(j)} = \vec{\mu}^{(j-1)} + \tau M^{-1}(\vec{\varphi}(t_j) - A\vec{\mu}^{(j)})$

Expl. Eul. (9.2.7.2): $\quad \vec{\mu}^{(j)} = \vec{\mu}^{(j-1)} + \tau_j \mathbf{M}^{-1}(\vec{\varphi}(t_{j-1}) - \mathbf{A}\vec{\mu}^{(j-1)})\,,$

Impl. Eul.: (9.2.7.3): $\quad \vec{\mu}^{(j)} = \underbrace{(\tau_j\mathbf{A} + \mathbf{M})}_{\text{s.p.d.}}{}^{-1}\left(\mathbf{M}\vec{\mu}^{(j-1)} + \tau_j\vec{\varphi}(t_j)\right)\,.$

• Runge-Kutta SSM :

Definition 7.3.3. 1. General Runge-Kutta method →

For coefficients $b_i, a_{ij} \in \mathbb{R}$, $c_i := \sum_{j=1}^{s} a_{ij}$, $i,j = 1,\dots,s$, $s \in \mathbb{N}$, the discrete evolution $\Psi^{s,t}$ of an s-stage Runge-Kutta single step method (RK-SSM) for the ODE $\dot{\mathbf{u}} = \mathbf{f}(t,\mathbf{u})$, is defined by

$$\mathbf{k}_i := \mathbf{f}(t + c_i\tau, \mathbf{u} + \tau\sum_{j=1}^{s} a_{ij}\mathbf{k}_j)\,, \quad i = 1,\dots,s \quad , \quad \Psi^{t,t+\tau}\mathbf{u} := \mathbf{u} + \tau\sum_{i=1}^{s} b_i\mathbf{k}_i\,.$$

Implicit increment equations

The $\mathbf{k}_i \in V_0$ are called increments.

$\vec{\kappa}_i \in \mathbb{R}^N$

$$\vec{\kappa}_i = M^{-1}\left(\vec{\varphi}(t + c_i\tau) - A\left(\vec{\mu}^{(j)} + \tau\sum_{j=1}^{s} a_{ij}\vec{\kappa}_j\right)\right)$$

$\Downarrow$

$$\vec{\kappa}_i \in \mathbb{R}^N:\quad \mathbf{M}\vec{\kappa}_i + \sum_{m=1}^{s}\tau a_{im}\mathbf{A}\vec{\kappa}_m = \vec{\varphi}(t_j + c_i\tau) - \mathbf{A}\vec{\mu}^{(j)}\,, \quad i = 1,\dots,s\,, \qquad (9.2.7.7)$$

$$\vec{\mu}^{(j+1)} = \vec{\mu}^{(j)} + \tau\sum_{m=1}^{s}\vec{\kappa}_m b_m\,. \qquad (9.2.7.8)$$

$\hat{=}$ $N\cdot s \times N\cdot s$ linear system of equations

Compact notation via Kronecker product

$$(\ .\ .7.7) \Leftrightarrow \underbrace{(\mathbf{I}_s \otimes \mathbf{M} + \tau\mathfrak{A} \otimes \mathbf{A})}_{\in \mathbb{R}^{Ns,Ns}}\begin{bmatrix}\vec{\kappa}_1\\ \vdots \\ \vec{\kappa}_s\end{bmatrix} = \begin{bmatrix}\vec{\varphi}(t_j + c_1\tau) - \mathbf{A}\vec{\mu}^{(j)}\\ \vdots \\ \vec{\varphi}(t_j + c_s\tau) - \mathbf{A}\vec{\mu}^{(j)}\end{bmatrix}\,. \qquad (9.2.7.9)$$

③ 9.2.7.2 Stability

Parabolic evolution problem in one spatial dimension (IBVP):

Exp 9.2.7.10

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in } [0,1]\times]0,1[\,, \tag{ .2.7.11}$$

$u(t,0) = u(t,1) = 0$ for $0 \le t \le 1$, $u(0,x) = \sin(\pi x)$ for $0 < x < 1$.

▶ exact solution $u(t,x) = \exp(-\pi^2 t)\sin(\pi x)$.

- Spatial finite element Galerkin discretization by means of linear finite elements ($V_{0,h} = \mathcal{S}^0_{1,0}(\mathcal{M})$) on equidistant mesh $\mathcal{M}$ with meshwidth $h := \frac{1}{N+1}$ → Section 2.3.
- $u_{N,0} := \mathsf{I}_1 u_0$ by linear interpolation on $\mathcal{M}$, see Section 3.3.1.
- Timestepping by explicit and implicit Euler method (7.1.7.2), (7.1.7.3) with uniform timestep $\tau := \frac{1}{M}$.

Monitored: $\text{error}^2 = \tau h \sum_{i=1}^{N}\sum_{j=1}^{M} |(u_h - u)(hi, j\tau)|^2$

Space-time (discrete) L^2-norm of error for explicit Euler timestepping:

$N\backslash M$	50	100	200	400	800	1600	3200
5	Inf	0.009479	0.006523	0.005080	0.004366	0.004011	0.003834
10	Inf	Inf	Inf	Inf	0.001623	0.001272	0.001097
20	Inf	Inf	Inf	Inf	Inf	Inf	0.000405
40	Inf	Inf	Inf	Inf	Inf	Inf	Inf
80	Inf	Inf	Inf	Inf	Inf	Inf	Inf
160	Inf	Inf	Inf	Inf	Inf	Inf	Inf
320	Inf	Inf	Inf	Inf	Inf	Inf	Inf

Inf → blow-up for small h_M, large τ

Space-time (discrete) L^2-norm of error for implicit Euler timestepping:

$N\backslash M$	50	100	200	400	800	1600	3200
5	0.007025	0.001828	0.000876	0.002257	0.002955	0.003306	0.003482
10	0.009641	0.004500	0.001826	0.000461	0.000228	0.000575	0.000749
20	0.010303	0.005175	0.002509	0.001149	0.000461	0.000116	0.000058
40	0.010469	0.005345	0.002681	0.001321	0.000634	0.000289	0.000116
80	0.010511	0.005387	0.002724	0.001364	0.000677	0.000332	0.000159
160	0.010521	0.005398	0.002734	0.001375	0.000688	0.000343	0.000170
320	0.010524	0.005400	0.002737	0.001378	0.000691	0.000346	0.000172

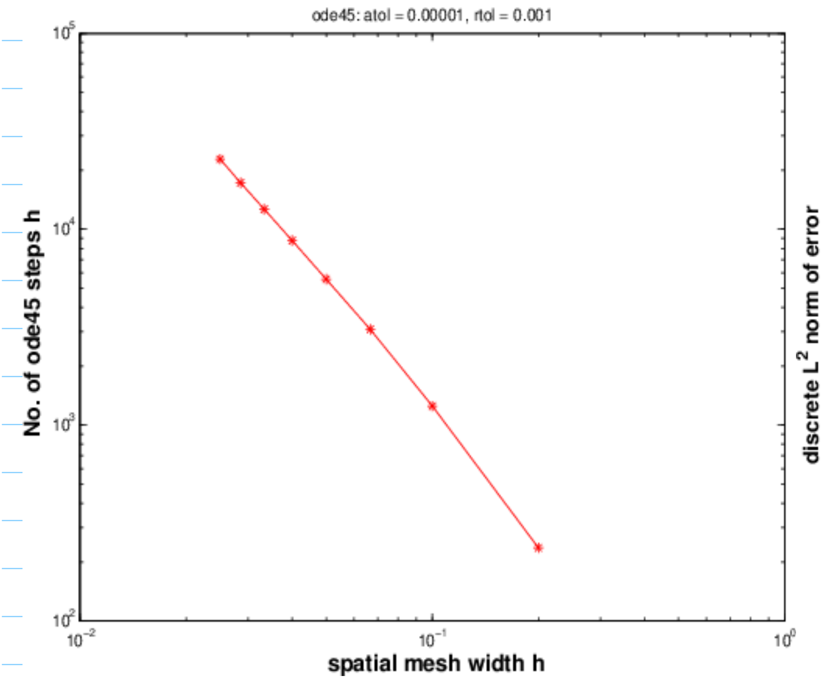
▷ unconditional stability

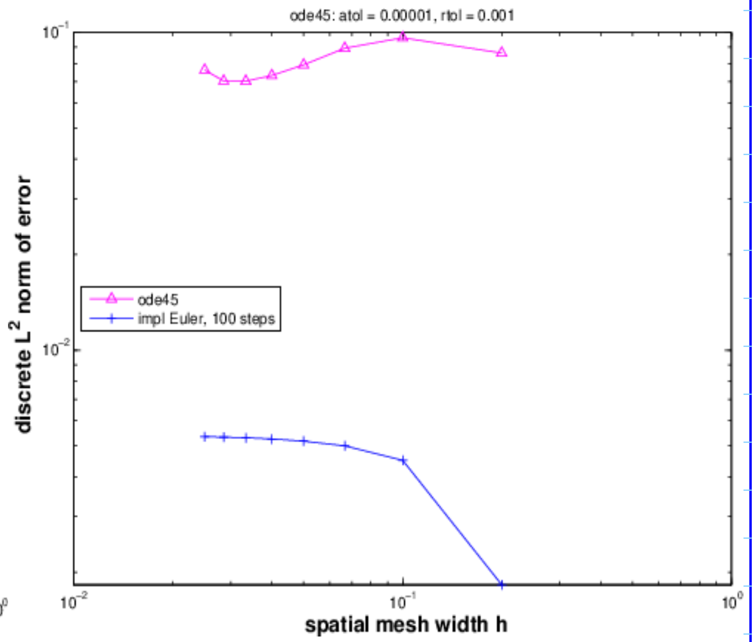
For explicit Euler timestepping we observe a glaring instability (exponential blow-up) in case of *large timestep combined with fine mesh.*

Implicit Euler timestepping incurs no blow-up for any combination of spatial and temporal mesh width.

Just a shortcoming of expl. Euler?

Exp 9.2.7.13: Adaptive explicit RK-SSM 4(5)





$M \approx h_m^{-2}$

(determined by stepsize control)

[MOL-ODE is stiff]

Implicit Euler superior!

§ 9.2.7.15: Analysis by diagonalization

$\dot{\underline{u}} = A\underline{u}$, $A \in \mathbb{R}^{N,N}$ diagonalizable

$$AT = TD \,, \quad D = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$$

$$\underline{w} = T^{-1}\underline{u} \;\Rightarrow\; \dot{\underline{w}} = D\underline{w} \quad : \text{decoupling}$$

$$\Leftrightarrow \quad \dot{w}_i = \lambda_i w_i$$

s.p.d. matrices $\in \mathbb{R}^{N,N}$

$$\mathbf{M}\left\{\tfrac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \,. \qquad (9.2.4.4)$$

Let $\vec{\varphi}_1, \dots, \vec{\varphi}_N \in \mathbb{R}^N$ denote the N linearly independent generalized eigenvectors satisfying

$$\mathbf{A}\vec{\psi}_i = \lambda_i \mathbf{M}\vec{\psi}_i \quad , \quad \vec{\psi}_j^{\top}\mathbf{M}\vec{\psi}_i = \delta_{ij} \,, \quad 1 \le i,j \le N \,. \qquad (9.2.7.16)$$

with positive eigenvalues $\lambda_i > 0$. Introducing the regular square matrices

$$\mathbf{T} = [\vec{\psi}_1, \dots, \vec{\psi}_N] \in \mathbb{R}^{N,N} \,, \qquad (9.2.7.17)$$

$$\mathbf{D} := \mathrm{diag}(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^{N,N} \,, \qquad (9.2.7.18)$$

we can rewrite (7.1.7.18) as

$$\mathbf{AT} = \mathbf{MTD} \quad , \quad \mathbf{T}^{\top}\mathbf{MT} = \mathbf{I} \,. \;\rightarrow\; (MT)^{-1} = T^{\top} \qquad (9.2.7.19)$$

New coefficient vector: $\vec{\eta}(t) = T^{-1}\vec{\mu}(t)$

MOL-ODE $\Rightarrow$ $T^{\top}\cdot \mid \; MT\dot{\vec{\eta}} + AT\vec{\eta} = \vec{\varphi}$

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$\triangleright$ $\underbrace{\dot{\vec{\eta}} = -D\vec{\eta} + T^T\vec{\varphi}(t)}$

Decoupled scalar ODEs

$$\dot{\eta}_i = -\lambda_i \eta_i + (T^T\vec{\varphi}(t))_i$$

[$\to$ Exponentially decaying solutions for $\vec{\varphi}$]

Diagonalizing explicit Euler method ($\vec{\varphi} \equiv 0$)

$$\vec{\mu}^{(j)} = \vec{\mu}^{(j-1)} - \tau \mathbf{M}^{-1}\mathbf{A}\vec{\mu}^{(j-1)} \quad \xrightarrow{\vec{\eta} := \mathbf{T}^\top \mathbf{M}\vec{\mu}} \quad \vec{\eta}^{(j)} = \vec{\eta}^{(j-1)} - \tau \mathbf{D}\vec{\eta}^{(j-1)},$$

$$\Updownarrow$$

$$\eta_i^{(j)} = \eta_i^{(j-1)} - \tau\lambda_i \eta_i^{(j-1)}$$

$$\eta_i^{(j)} = \eta_i^{(j-1)} - \tau\lambda_i \eta_i^{(j-1)} \quad \Rightarrow \quad \boxed{\eta_i^{(j)} = (1-\tau\lambda_i)^j \eta_i^{(0)}}. \qquad (9.2.7.24)$$

$$|1-\tau\lambda_i| < 1 \quad \Leftrightarrow \quad \lim_{j\to\infty} \eta_i^{(j)} = 0.$$

$$|1-\tau\lambda_i| > 1 \quad \Rightarrow \quad \text{blow-up}$$

The condition $|1-\tau\lambda_i| < 1$ enforces a

timestep size constraint: $\tau < \dfrac{2}{\lambda_i} \quad \forall i$

If $\exists \lambda_i : |1-\tau\lambda_i| > 1 \Rightarrow$ Blow-up of $\vec{\mu}^{(j)}$

How do λ_i behave for FE-Galerkin:

$$\underset{\substack{\uparrow \\ \text{stiffness}}}{A}\vec{\Psi}_i = \lambda_i \underset{\substack{\uparrow \\ \text{mass matrix}}}{M}\vec{\Psi}_i$$

Exp 9.2.7.26: Linear Lagr. FE, h-refinement

Bilinear forms associated with parabolic IBVP and homogeneous Dirichlet boundary conditions

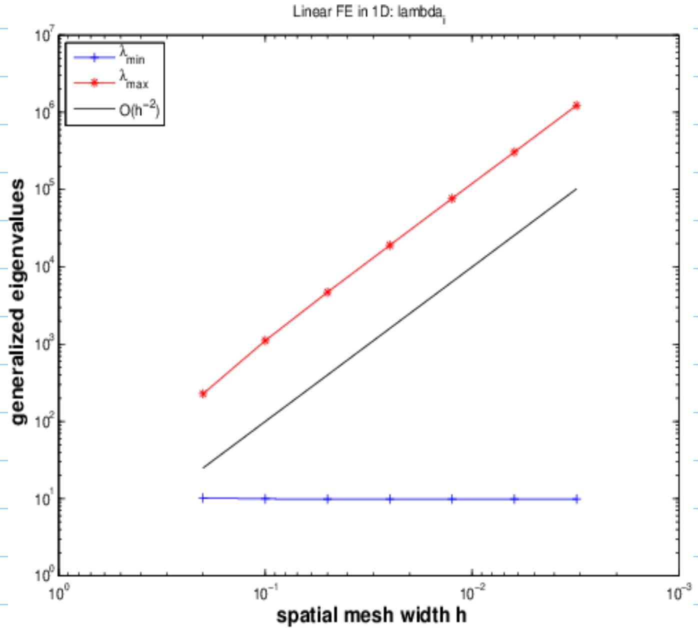
$$\mathrm{a}(u,v) = \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x \;, \quad \mathrm{m}(u,v) = \int_\Omega u(x)v(x)\,\mathrm{d}x \;, \quad u,v \in H_0^1(\Omega).$$

Linear finite element Galerkin discretization, see Section 2.3 for 1D, and Section 2.4 for 2D.

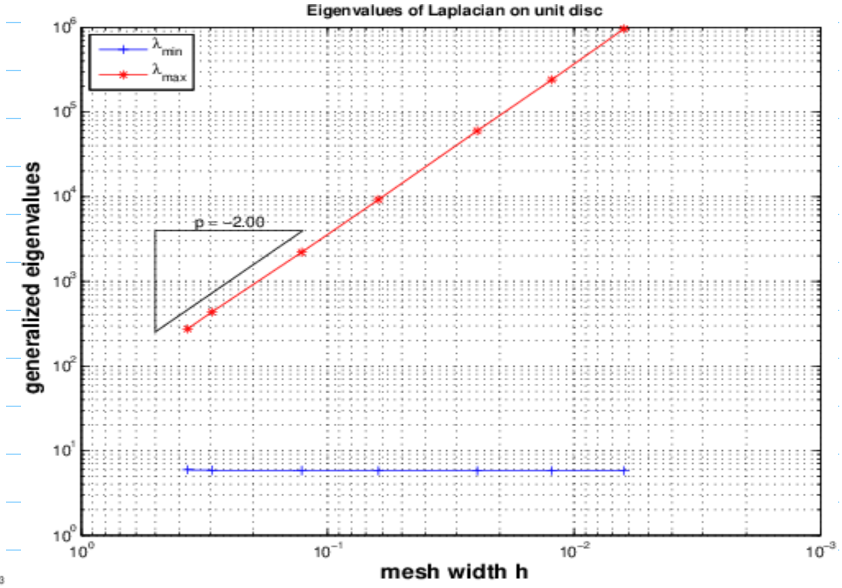
Numerical experiments in 1D & 2D:

- $\Omega =]0,1[$, equidistant meshes $\to$ Exp. 7.1.7.10
- "disk domain" $\Omega = \{x \in \mathbb{R}^2 : \|x\| < 1\}$, sequence of regularly refined meshes.

Monitored: largest and smallest generalized eigenvalue



$\Omega =]0,1[$



$\Omega = \{x \in \mathbb{R}^2 : \|x\| < 1\}$

Observation : λ_{min} virtually indep. of $h_{\mathcal{M}}$

$\lambda_{max} = O(h_{\mathcal{M}}^{-2})$ for $h_{\mathcal{M}} \to 0$

Lemma 9.2.7.30. Behavior of of generalized eigenvalues

Let $\mathcal{M}$ be a simplicial mesh and $\mathbf{A}$, $\mathbf{M}$ denote the Galerkin matrices for the bilinear forms $\mathsf{a}(u,v) = \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}\boldsymbol{x}$ and $\mathsf{m}(u,v) = \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, respectively, and $V_{0,h} := \mathcal{S}^0_{p,0}(\mathcal{M})$. Then the smallest and largest generalized eigenvalues of $\mathbf{A}\vec{\mu} = \lambda\mathbf{M}\vec{\mu}$, denoted by $\lambda_{\min}$ and $\lambda_{\max}$, satisfy

$$\frac{1}{\operatorname{diam}(\Omega)^2} \leq \lambda_{\min} \leq C \quad , \quad \lambda_{\max} \geq C h_{\mathcal{M}}^{-2} ,$$

where the "generic constants" (→ Rem. 3.3.5.10) depend only on the polybomial degree p, the domain Ω, and the shape regularity measure $\rho_{\mathcal{M}}$.

Exp. Euler in FE-setting, h-ref : $\tau \lesssim C h_{\mathcal{M}}^2$

[also observed with ode45]

§9.2.7.35 : Implicit Euler : Diagonalization

$$\vec{\mu}^{(j)} = \vec{\mu}^{(j-1)} - \tau\mathbf{M}^{-1}\mathbf{A}\vec{\mu}^{(j)} \quad \xrightarrow{\vec{\eta}:=\mathbf{T}^\top\mathbf{M}\vec{\mu}} \quad \vec{\eta}^{(j)} = \vec{\eta}^{(j-1)} - \tau\mathbf{D}\vec{\eta}^{(j)} ,$$

$$\eta_i^{(j)} = \eta_i^{(j-1)} - \tau\lambda_i\eta_i^{(j)} \quad \Rightarrow \quad \boxed{\eta_i^{(j)} = \left(\frac{1}{1+\tau\lambda_i}\right)^j \eta_i^{(0)}} . \tag{9.2.7.36}$$

$$\left[\left|\frac{1}{1+\tau\lambda_i}\right| < 1 \text{ and } \lambda_i > 0 \Rightarrow \right] \quad \lim_{j\to\infty} \eta_i^{(j)} = 0 \quad \forall\tau > 0 . \tag{9.2.7.37}$$

unconditional stability

§9.2.7.38 : Diagonalization of RK-SSM ($\vec{\varphi} \equiv 0$)

[MOL-ODE]

$$\begin{array}{ccc}
\mathbf{M}\frac{d}{dt}\vec{\mu} + \mathbf{A}\mu = 0 & \xrightarrow{\text{transformation } \vec{\eta} = \mathbf{T}^T\mathbf{M}\vec{\mu}} & \frac{d}{dt}\eta_i = -\lambda_i\eta_i \,, i = 1,\ldots,N \\
\Big\downarrow \text{RK-SSM} & & \Big\downarrow \text{RK-SSM} \\
\vec{\mu}^{(j)} = \Psi^{t,t+\tau}\vec{\mu}^{(j-1)} & \xrightarrow{\text{transformation } \vec{\eta}^{(j)} = \mathbf{T}^T\mathbf{M}\vec{\mu}^{(j)}} & \eta_i^{(j)} = \widetilde{\Psi}_i^{t,t+\tau}\eta_i^{(j-1)} \,, i = 1,\ldots,N .
\end{array} \tag{9.2.7.41}$$

DE for MOL-ODE

Discrete evolution for RK-SSM applied to $\dot{u} = -\lambda_i u$ [scalar ODE]

▷ Uncond. stab. of RK-SSM for MOL-ODE

⇕

// of RK-SSM for $\dot{u} = -\lambda u$, $\lambda > 0$

Unconditional stability of single step methods

Necessary condition for *universal unconditional stability* of a single step method for semi-discrete parabolic evolution problem (7.1.4.4)("method of lines"):

The discrete evolution $\Psi^\tau_\lambda : \mathbb{R} \mapsto \mathbb{R}$ of the single step method applied to the scalar ODE $\dot{u} = -\lambda u$ satisfies

$$\lambda > 0 \quad \Rightarrow \quad \lim_{j\to\infty} (\Psi^\tau_\lambda)^j u_0 = 0 \quad \forall u_0 \in \mathbb{R}, \quad \forall\tau > 0 . \tag{9.2.7.43}$$

RK-SSM for $\dot{u} = -\lambda u \Rightarrow u^{(j)} = S(-\lambda\tau)\, u^{(j-1)}$

$$\begin{array}{c|c} c & \mathfrak{A} \\ \hline & b^T \end{array}$$

Stability functions

$$\Psi_\lambda^{t,t+\tau} u = S(-\lambda\tau)u\,, \tag{6.2.7.47a}$$

with rational stability function

$$S(z) = 1 + z\mathbf{b}^\top (\mathbf{I} - z\mathfrak{A})^{-1}\mathbf{1} = \frac{\det(\mathbf{I} - z\mathfrak{A} + z\mathbf{1}\mathbf{b}^\top)}{\det(\mathbf{I} - z\mathfrak{A})}\,, \quad \mathbf{1} = [1,\dots,1]^\top \in \mathbb{R}^s\,. \tag{6.2.7.47b}$$

↳ a rational function in z

RK-SSM uncond. stable for MOL-ODE if $|S(z)| \leq 1 \quad \forall z < 0$

Even desirable $|S(z)| \ll 1$ for $z < 0$, $|z| \gg 1$

[fast exponential decay]

Definition 6.2.7.46. L(π)-stability

A Runge-Kutta single-step method satisfying (7.1.7.45) is called L(π)-stable, if its stability function $S(z)$ according to (7.1.7.47b) satisfies

(i) $|S(z)| < 1$ for all $z < 0$, and

(ii) "$S(-\infty)$" $:= \lim\limits_{z\in\mathbb{R}\to-\infty} S(z) = 0$.

Ex 9.2.7.49:

$$\begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array} \tag{6.2.7.52}$$

RADAU-3 scheme (order 3)

$$\begin{array}{c|cc} \lambda & \lambda & 0 \\ 1 & 1-\lambda & \lambda \\ \hline & 1-\lambda & \lambda \end{array}\,, \quad \lambda := 1 - \tfrac{1}{2}\sqrt{2}\,, \tag{6.2.7.53}$$

SDIRK-2 scheme (order 2)

Note: Necessarily implicit

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Review questions 9.2.7.52

[Tackle without relying on notes]

A :

The implicit 2-stage Gauss-Radau method is described by the Butcher scheme

$$\begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array} .$$

Describe the implementation of a single step of this method for the method-of-lines ODE

$$\frac{d}{dt}\vec{\mu}(t) + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) . \qquad (6.2.4.4)$$

Definition 6.2.6.31. General Runge-Kutta method

For coefficients $b_i, a_{ij} \in \mathbb{R}$, $c_i := \sum_{j=1}^{s} a_{ij}$, $i, j = 1, \ldots, s$, $s \in \mathbb{N}$, the discrete evolution $\mathbf{\Psi}^{s,t}$ of an s-stage Runge-Kutta single step method (RK-SSM) for the ODE $\dot{\mathbf{u}} = \mathbf{f}(t, \mathbf{u})$, is defined by

$$\mathbf{k}_i := \mathbf{f}(t + c_i\tau, \mathbf{u} + \tau \sum_{j=1}^{s} a_{ij}\mathbf{k}_j) , \quad i = 1, \ldots, s \quad , \quad \mathbf{\Psi}^{t,t+\tau}\mathbf{u} := \mathbf{u} + \tau \sum_{i=1}^{s} b_i \mathbf{k}_i .$$

The $\mathbf{k}_i \in V_0$ are called increments.

Butcher-scheme notation:

$$\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^T \end{array} \quad \hat{=} \quad \begin{array}{c|ccccc} c_1 & a_{11} & a_{12} & \cdots & \cdots & a_{1s} \\ c_2 & a_{21} & \ddots & & & a_{2s} \\ \vdots & \vdots & & \ddots & & \vdots \\ c_s & a_{s1} & \cdots & & & a_{ss} \\ \hline & b_1 & b_2 & \cdots & \cdots & b_s \end{array} \quad , \quad \mathbf{c}, \mathbf{b} \in \mathbb{R}^s, \quad \mathfrak{A} \in \mathbb{R}^{s,s} . \qquad (6.2.6.33)$$

B :

How does blow-up of a timestepping scheme for the method-of-lines ODE (6.2.4.4) manifest itself.

C :

Explain, why it is enough to understand the behavior of a Runge-Kutta single-step method for the scalar IVP

$$\dot{y} = \lambda y \quad , \quad y(0) = 1 \, , \quad \lambda \in \mathbb{R} \, ,$$

in order to predict its stability properties when applied to the method-of-lines ODE (6.2.4.4).

D :

We discretize the parabolic initial-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\boldsymbol{\alpha}(\boldsymbol{x}) \,\mathbf{grad}\, u) = 0 \quad &\text{in} \quad \Omega \times [0, T] \, , \\ u = 0 \quad &\text{on} \quad \partial\Omega \times [0, T] \, , \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad &\text{on} \quad \Omega \, , \end{aligned}$$

in space by means of quadratic Lagrangian finite elements on sequencies of triangular meshes created by uniform regular refinement. Method-of-lines timestepping relies on an *explicit* Runge-Kutta single-step method with uniform timestep $\tau > 0$.
What relationship between timestep τ and meshwidth $h_{\mathcal{M}}$ has to be imposed in order to ensure stability of the resulting fully discrete evolution?

E :

Consider the evolution problem

$$t \in]0,T[\mapsto u(t) \in H^1(\Omega): \quad \frac{d}{dt}\int_{\partial\Omega} u(t) v \,\mathrm{d}S + \int_{\Omega} \mathbf{grad}\, u(t) \cdot \mathbf{grad}\, v \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall v \in H^1(\Omega)\,.$$

We perform spatial finite element Galerkin semi-discretization based on $\mathcal{S}_1^0(\mathcal{M})$ in the spirit of the method of lines.

1. Which problem does the application of explicit Runge-Kutta timestepping face?
2. Show that implicit Euler timestepping is feasible.

F:

Show that Crank-Nicolson timestepping for the MOL-ODE

$$\mathbf{M}\left\{\tfrac{d}{dt}\vec{\boldsymbol{\mu}}(t)\right\} + \mathbf{A}\vec{\boldsymbol{\mu}}(t) = \vec{\boldsymbol{\varphi}}(t)$$

$$\Downarrow$$

$$\mathbf{M}\frac{\vec{\boldsymbol{\mu}}^{(j)} - \vec{\boldsymbol{\mu}}^{(j-1)}}{\tau} = -\tfrac{1}{2}\mathbf{A}\left(\vec{\boldsymbol{\mu}}^{(j)} + \vec{\boldsymbol{\mu}}^{(j-1)}\right) + \tfrac{1}{2}\left(\vec{\boldsymbol{\varphi}}(t_j) + \vec{\boldsymbol{\varphi}}(t_{j-1})\right)\,, \tag{6.2.7.5}$$

for a standard parabolic evolution problem with s.p.d. bilinear forms $\mathsf{m}(\cdot,\cdot)$ and $\mathsf{a}(\cdot,\cdot)$ is *unconditionally stable*.