ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 9.2.8: Fully Discrete Method of Lines: Convergence

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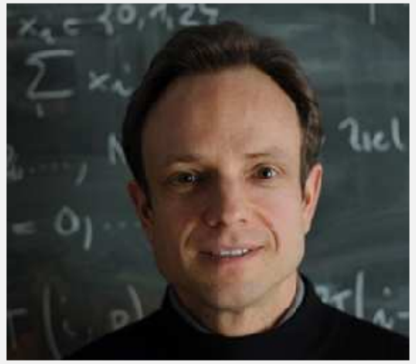

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Dependency. [Lecture → Section 9.2.4] and [Lecture → Section 3.3.5].

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:

Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11

Trailing digits in labels are not affected.

Duration: 35 minutes

IX. Second-Order Linear Evolution Problems

9.2. Parabolic Initial-Boundary-Value Problems

9.2.8 Fully Discrete Method of Lines: Convergence

Heat equation:

$$\frac{\partial}{\partial t}(\rho(\boldsymbol{x})u) - \operatorname{div}(\kappa(\boldsymbol{x})\,\mathbf{grad}\,u) = f \quad \text{in} \quad \widetilde{\Omega} := \Omega\times]0,T[\,, \qquad (9.2.1.6)$$

$$u(\boldsymbol{x},t) = g(\boldsymbol{x},t) \quad \text{for} \quad (\boldsymbol{x},t)\in\partial\Omega\times]0,T[\,, \qquad (9.2.1.7)$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x}\in\Omega\,. \qquad (9.2.1.9)$$

Spatial variational formulation: $[g=0]$

Spatial variational form of (7.1.1.5)–(7.1.1.7): seek $t\in]0,T[\mapsto u(t)\in H_0^1(\Omega)$

$$\int_\Omega \rho(\boldsymbol{x})\dot{u}(t)v\,\mathrm{d}\boldsymbol{x} + \int_\Omega \kappa(\boldsymbol{x})\,\mathbf{grad}\,u(t)\cdot\mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x},t)v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall v\in H_0^1(\Omega)\,, \qquad (9.2.2.4)$$

$$u(0) = u_0 \in H_0^1(\Omega)\,. \qquad (9.2.2.5)$$

② Abstract:

$$t \in]0,T[\mapsto u(t) \in V_0 \quad : \quad \begin{cases} \frac{d}{dt}\mathsf{m}(u(t),v) + \mathsf{a}(u(t),v) = \ell(t)(v) \quad \forall v \in V_0\,, \\ u(0) = u_0 \in V_0\,. \end{cases} \tag{9.2.2.9}$$

1st step: replace V_0 with a finite dimensional subspace $V_{0,h}$, $N := \dim V_{0,h} < \infty$

▶ (Spatially) discrete parabolic evolution problem

$$t \in]0,T[\mapsto u_h(t) \in V_{0,h} \quad : \quad \begin{cases} \mathsf{m}(\dot{u}_h(t),v_h) + \mathsf{a}(u_h(t),v_h) = \ell(t)(v_h) \quad \forall v_h \in V_{0,h}\,, \\ u_h(0) = \text{projection/interpolant of } u_0 \text{ in } V_{0,h}\,. \end{cases} \tag{9.2.4.1}$$

Method-of-lines ordinary differential equation

Combining (7.1.4.1) and (7.1.4.2) we obtain

$$(7.1.4.1) \quad \Rightarrow \quad \begin{cases} \mathbf{M}\left\{\frac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \quad \text{for } 0 < t < T\,, \\ \vec{\mu}(0) = \vec{\mu}_0\,. \end{cases} \tag{9.2.4.4}$$

with
- ▷ s.p.d. stiffness matrix $\mathbf{A} \in \mathbb{R}^{N,N}$, $(\mathbf{A})_{ij} := \mathsf{a}(b_h^j, b_h^i)$ (independent of time),
- ▷ s.p.d. mass matrix $\mathbf{M} \in \mathbb{R}^{N,N}$, $(\mathbf{M})_{ij} := \mathsf{m}(b_h^j, b_h^i)$ (independent of time),
- ▷ source (load) vector $\vec{\varphi}(t) \in \mathbb{R}^N$, $(\vec{\varphi}(t))_i := \ell(t)(b_h^i)$ (time-dependent),
- ▷ $\vec{\mu}_0 \mathrel{\hat=}$ coefficient vector of a projection of u_0 onto $V_{0,h}$.

Note:

(7.1.4.4) is an ordinary differential equation (ODE) for $t \mapsto \vec{\mu}(t) \in \mathbb{R}^N$

Definition 7.3.3.1. General Runge-Kutta method →

For coefficients $b_i, a_{ij} \in \mathbb{R}$, $c_i := \sum_{j=1}^{s} a_{ij}$, $i,j = 1,\dots,s$, $s \in \mathbb{N}$, the discrete evolution $\Psi^{s,t}$ of an s-stage Runge-Kutta single step method (RK-SSM) for the ODE $\dot{\mathbf{u}} = \mathbf{f}(t,\mathbf{u})$, is defined by

$$\mathbf{k}_i := \mathbf{f}(t + c_i\tau, \mathbf{u} + \tau\sum_{j=1}^{s} a_{ij}\mathbf{k}_j)\,, \quad i = 1,\dots,s \quad , \quad \Psi^{t,t+\tau}\mathbf{u} := \mathbf{u} + \tau\sum_{i=1}^{s} b_i\mathbf{k}_i\,.$$

The $\mathbf{k}_i \in V_0$ are called increments.

[Asymptotic alg. cvg. with order q for $\tau \to 0$]

- Explicit RK-SSM for MOL-ODE & fixed-degree Lagr. FE
  ▷ Stability-induced timestep constraint $\tau \leq C h_M^2$
- $L(\pi)$-stable RK-SSM : unconditionally stable
  ↳ Use these! unknown constant

Why: Unknown stability bound!

Another reason from convergence theory:

h-refinement setting, two discretization parameters

meshwidth $h_M \longrightarrow 0$
timestep $\tau \longrightarrow 0$ } asymptotic perspective

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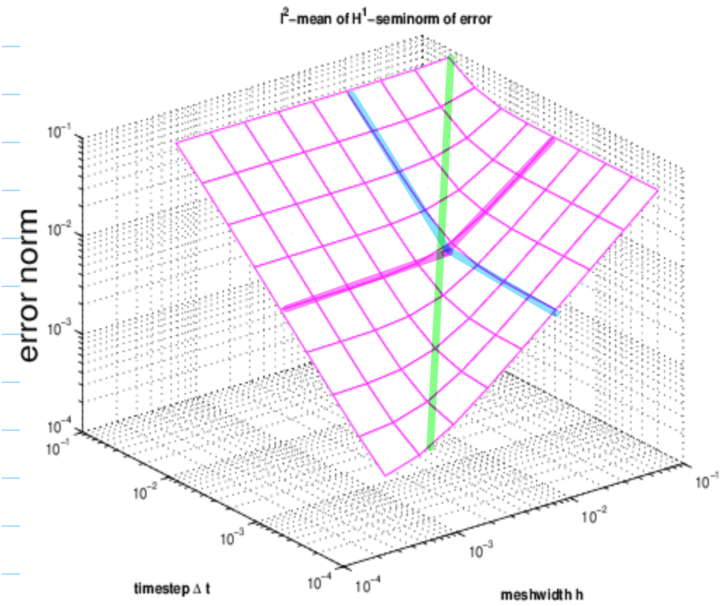
Exp. 9.2.8.1

- 1D parabolic evolution problem: $\frac{d}{dt}u - u'' = f(t,x)$ on $]0,1[\times]0,1[$
- exact solution $u(x,t) = (1+t^2)e^{-\pi^2 t}\sin(\pi x)$, source term accordingly
- Linear finite element Galerkin discretization equidistant mesh, see Section 2.3, $V_{0,h} = \mathcal{S}^0_{1,0}(\mathcal{M})$, ✦ piecewise linear spatial approximation of source term $f(x,t)$
- implicit Euler timestepping ($\rightarrow$ Ex. 7.1.7.1) with uniform timestep $\tau > 0$

uncond. stable 1-st order

Monitored: error norm $\left(\tau \sum_{j=1}^{M} |u - u_h(\tau j)|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}$.: L^2 in time, H^1 in space



◁ $h_{\mathcal{M}}$- and τ-dependence of error norm

Observation:

τ small: error norm $\approx h_{\mathcal{M}}$
$h_{\mathcal{M}}$ small: error norm $\approx \tau$
} alg. cvg. rate 1

The error seems to behave like

$$\text{error norm} \approx C_1 h_{\mathcal{M}} + C_2\tau\,. \tag{9.2.8.2}$$

total error = spatial error + temporal error

"Meta-theorem" 9.2.8.5. Convergence of solutions of fully discrete parabolic evolution problems

Assume that

- *the solution of the parabolic IBVP (7.1.1.5)–(7.1.1.8) is "sufficiently smooth" (both in space and time),*
- *its spatial Galerkin finite element discretization relies on degree p Lagrangian finite elements ($\rightarrow$ Section 2.6) on uniformly shape-regular families of meshes,*
- *timestepping is based on an $L(\pi)$-stable single step method of order q with uniform timestep $\tau > 0$.*

Then we can expect an asymptotic behavior of the total discretization error according to

$$\left(\tau \sum_{j=1}^{M} |u - u_h(\tau j)|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \leq C\left(h^p_{\mathcal{M}} + \tau^q\right), \tag{9.2.8.6}$$

where $C > 0$ must not depend on $h_{\mathcal{M}}, \tau$.

unknown generic constant   spatial error   temporal error

▷ Tells trend of the error for $h_m, \tau \rightarrow 0$

Assume: Bound sharp: spatial error $\approx C h^p_{\mathcal{M}}$
temporal error $\approx C\tau^q$

Goal: reduction of total error by a factor of $\varsigma > 1$

$h_m \longrightarrow h_m \cdot \varsigma^{-1/p}$
$\tau \longrightarrow \tau \cdot \varsigma^{-1/q}$
} : error equilibration principle

Refinement for fully discrete parabolic evolution problems

Guideline: spatial and temporal resolution have to be adjusted in tandem

What if timestepping scheme is only conditionally stable:

$$\tau \leq C h_m^2$$

$h_m \longrightarrow h_m \cdot \rho^{-1/p}$ to reduce spatial error

$\tau \longrightarrow \tau \cdot \rho^{-2/p}$

If $1/q < 2/p$ $\Rightarrow$ reduction of τ required by stability-induced timestep constraint is more severe than the reduction required by control of temporal error

$\longrightarrow$ inefficient: small timesteps but no commensurate gain in accuracy

$q = 4$ for classical RK-SSM

$\Rightarrow$ $p = 9$ required for efficient ("balanced") fully discrete MOL.

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Review questions 9.2.8.14

[Answer without looking at the lecture document]

A:

How will the assertion of Thm. 6.2.8.5 will probably have to be altered in case we face $u(t) \in H^m(\Omega)$, $m \geq 2$, but $u(t) \notin H^{m+1}(\Omega)$ for all times t.

Theorem 6.2.8.5. Convergence of solutions of fully discrete parabolic evolution problems

Assume that

- *the solution of the parabolic IBVP* (6.2.1.6)–(6.2.1.9) *is "sufficiently smooth" (both in space and time),*
- *its spatial Galerkin finite element discretization relies on degree p Lagrangian finite elements ($\rightarrow$ Section 2.6) on uniformly shape-regular families of meshes,*
- *timestepping is based on an $L(\pi)$-stable single step method of order q with uniform timestep $\tau > 0$.*

Then we can expect an asymptotic behavior of the total discretization error according to

$$\left(\tau \sum_{j=1}^{M} |u - u_h(\tau j)|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \leq C(h^p_{\mathcal{M}} + \tau^q) , \tag{6.2.8.6}$$

where $C > 0$ must not depend on $h_{\mathcal{M}}$, τ.

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