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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

# Course Video
# Section 9.3.1: Models for Vibrating Membranes

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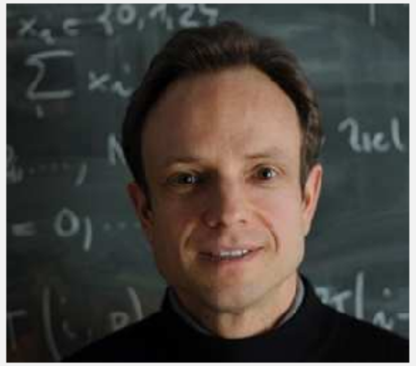


**Prerequisites.**
- Basic knowledge about elementary mechanics

**Dependency.** [Lecture → Section 1.2.1] and [Lecture → Section 9.1]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

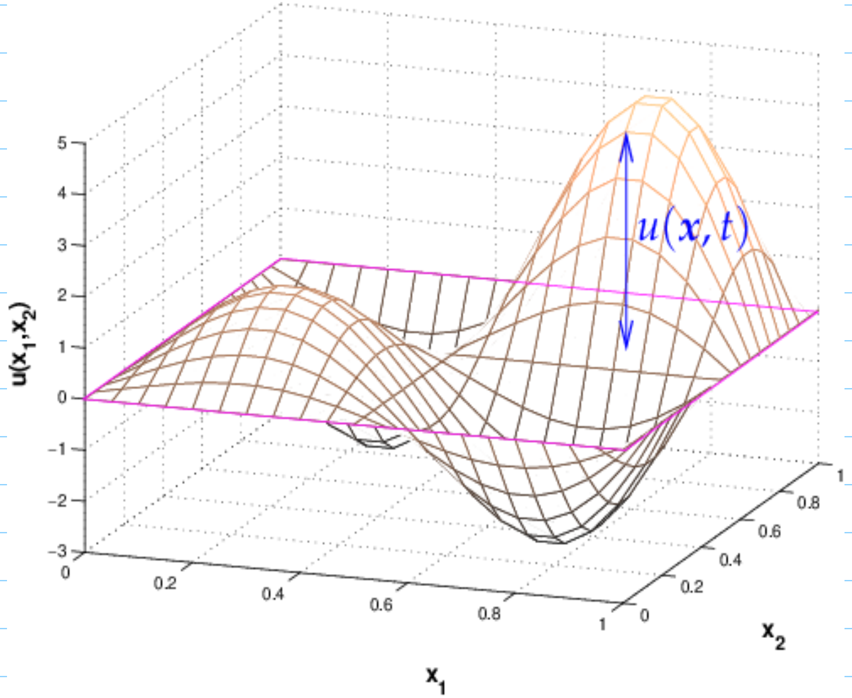
Duration: 24 minutes

# VI. Second-Order Linear Evolution Problems

## 9.3 Linear Wave Equations

↳ hyperbolic evolution equation

### 9.3.1 Models for Vibrating Membranes




Taut membrane $\leftrightarrow u : \Omega \mapsto \mathbb{R}$

◁ 2D graph model for shape of membrane

◁ Configuration space $V = H_0^1(\Omega)$ (flat frame)

# Stationary weak formulation of membrane model

$$u \in V: \quad \int_\Omega \sigma(\boldsymbol{x})\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \quad \forall v \in H_0^1(\Omega)\,, \tag{6.3.1.2}$$

where $f:\Omega \mapsto \mathbb{R}$ $\hat{=}$ density of vertical force,
$\sigma:\Omega \mapsto \mathbb{R}$ $\hat{=}$ uniformly positive stiffness coefficient (characteristic of material of the membrane).

Extend to dynamic model : $u = u(x,t)$

Moving membrane (without load) feels an inertia force

$$f(x,t) = -\rho(x)\frac{\partial^2 u}{\partial t^2}(x,t)$$

force density     mass density     acceleration

▶ Homogeneous linear wave equation in variational form (Dirichlet boundary conditions):

$$u \in V(t): \quad \int_\Omega \rho(\boldsymbol{x}) \cdot \frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t)\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \sigma(\boldsymbol{x})\,\mathbf{grad}\,u(\boldsymbol{x},t)\cdot\mathbf{grad}\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall v \in H_0^1(\Omega) \tag{6.3.1.7}$$

mass density     stiffness

$u : [0,T] \to V(t)$

$$\boxed{u \in V(t): \quad \mathsf{m}(\ddot{u},v) + \mathsf{a}(u,v) = 0 \quad \forall v \in V_0}\,. \tag{6.3.1.8}$$

where $V(t) := \{v : ]0,T[ \mapsto H^1(\Omega) : v(\boldsymbol{x},t) = g(\boldsymbol{x},t) \text{ for } \boldsymbol{x} \in \partial\Omega, 0 < t < T\}$
(with continuous time-dependent Dirichlet data $g : \partial\Omega\times]0,T[ \mapsto \mathbb{R}$.)

[ $a(\cdot,\cdot)$ , $m(\cdot,\cdot)$ ≙ the same as in parabolic case ]

(7.2.1.10) Lemma 1.5.3.4 $\Rightarrow$ $\boxed{\rho(\boldsymbol{x})\dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}(\sigma(\boldsymbol{x})\,\mathbf{grad}\,u) = 0 \quad \text{in } \tilde{\Omega}} = \Omega \times ]0,T[$ (6.3.1.11)

Linear wave equation

Extension : replace $0$ w/ $f(x,t)$ on r.h.s.

(As in the case of the heat equation)

- Spatial boundary conditions : "nothing new"

On $\partial\Omega\times]0,T[$ we can impose any of the boundary conditions discussed in Section 1.7:
- Dirichlet boundary conditions $u(\boldsymbol{x},t) = g(\boldsymbol{x},t)$ (membrane attached to frame),
- Neumann boundary conditions $\mathbf{j}(\boldsymbol{x},t)\cdot\boldsymbol{n} = 0$ (free boundary, Ex. 1.5.3.11)
- radiation boundary conditions $\mathbf{j}(\boldsymbol{x},t)\cdot\boldsymbol{n} = \Psi(u(\boldsymbol{x},t))$,

and any combination of these as discussed in Ex. 1.7.0.10, yet, *only one* of them at any part of $\partial\Omega\times]0,T[$, see Rem. 1.7.0.9.

- Initial conditions

Wave equation = 2nd-order in time

[ 2nd-order ODE : $\ddot{u} = g(t,u)$ )

[ $v := \dot{u}$ ]
velocity

$$\Leftrightarrow \quad \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ g(t,u) \end{bmatrix}$$

$\cancel{y}$ $\mathbf{f}(t,\mathbf{y})$

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▷ Initial condition : $u(0)$, $v(0) = \dot{u}(0)$

For LWE :
$$u(x,0) = u_0(x)$$
$$\frac{\partial u}{\partial t}(x,0) = v_0(x)$$

LWE → 1st-order in time system of PDEs

Additional unknown: velocity $v(\boldsymbol{x},t) = \frac{\partial u}{\partial t}(\boldsymbol{x},t)$

$$\rho(\boldsymbol{x})\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(\sigma(\boldsymbol{x})\operatorname{\mathbf{grad}} u) = 0 \quad \blacktriangleright \quad \boxed{\begin{cases} \dot{u} = v\,, \\ \rho(\boldsymbol{x})\dot{v} = \operatorname{div}(\sigma(\boldsymbol{x})\operatorname{\mathbf{grad}} u) \end{cases} \text{ in } \widetilde{\Omega}} \tag{6.3.1.18}$$

with initial conditions

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad , \quad v(\boldsymbol{x},0) = v_0(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \Omega\,. \tag{6.3.1.19}$$

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# Review questions 9.3.1.20

[ Answer without aids ]

A:

The movement of a taut membrane also subject to friction can be modeled by an additional vertical force density proportional to the local velocity:

$$f(\boldsymbol{x}) = -\rho(\boldsymbol{x})\frac{\partial^2 u}{\partial t^2} - \eta(\boldsymbol{x})\frac{\partial u}{\partial t} \,,$$

with uniformly positive friction coefficient $\epsilon = \epsilon(\boldsymbol{x})$. Give the spatial variational formulation for a membrance model with friction.

**Hint.** The dynamic membrane model with a generic force density $f = f(\boldsymbol{x})$ is:

$$u \in V: \quad \int_\Omega \sigma(\boldsymbol{x})\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v \,\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \quad \forall v \in H_0^1(\Omega)\,. \tag{6.3.1.2}$$

B:

Which initial-boundary value problem describes the frictionless movement of a membrane clamped to a flat and level frame on three sides of a square, but free on the fourth side under the influence of gravity?

C:

The proagation of sound in space can be described by the following first-order system of linear partial differential equations:

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0}\,\mathbf{grad}\,p = \mathbf{0}\,, \quad \frac{\partial \rho}{\partial t} + \rho_0\,\mathrm{div}\,\mathbf{v} = 0\,, \quad \frac{\partial \rho}{\partial t} - \frac{1}{c^2}\frac{\partial p}{\partial t} = 0\,.$$

Here $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t)$ is the velocity field ($[\mathbf{v}] = \mathrm{ms}^{-1}$), $p = p(\boldsymbol{x}, t)$ the pressure field ($[p] = \mathrm{Nm}^{-2}$), $\rho_0 = \rho_0(\boldsymbol{x})$ a uniformly positive density ($[\rho_0] = \mathrm{krm}^{-3}$), and $c = c(\boldsymbol{x})$ the local speed of sound ($[c] = \mathrm{ms}^{-1}$).

1. Derive a second-order PDE governing the evolution of the pressure field.
2. At hard walls we have $\mathbf{v}\cdot\boldsymbol{n} = 0$, where $\boldsymbol{n}$ is a unit vector normal to the wall. Which spatial boundary conditions does this entail for the second-order PDE?

D:

For symmetric positive definite matrices $\mathbf{M}, \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n,n}$ convert the second-order ODE for $\vec{\mu} : [0, T] \to \mathbb{R}^n$

$$\mathbf{M}\frac{\partial^2 \vec{\mu}}{\partial t^2} + \mathbf{B}\frac{\partial \vec{\mu}}{\partial t} + \mathbf{A}\vec{\mu} = \mathbf{0}$$

into an equivalent first-order ODE in the standard form $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$.

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