ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 9.3.2: Wave Propagation

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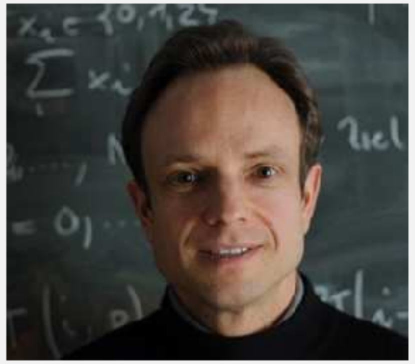
(C) Seminar für Angewandte Mathematik, ETH Zürich



Dependency. [Lecture → Section 9.3.1]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

Duration: 33 minutes

VI. Second-Order Linear Evolution Problems

9.3. Linear Wave Equations

9.3.2. Wave Propagation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(\sigma(x)\,\mathbf{grad}\,u) = 0 \quad \text{in} \quad \tilde{\Omega} := \Omega\times]0,T[\tag{6.3.1.11}$$

+ spatial boundary conditions & in initial conditions $\begin{cases} u(x,0) = u_0(x)\,, \\ \frac{\partial u}{\partial t}(x,0) = v_0(x)\,, \end{cases} \quad x\in\Omega\,.$

1D model Cauchy problem: $\Omega = \mathbb{R}$ (no spatial b.c.)

$$c>0:\quad \frac{\partial^2 u}{\partial t^2} - \underbrace{c^2}_{\text{wave speed}}\frac{\partial^2 u}{\partial x^2} = \overset{\text{no forcing}}{0}\;,\quad \underbrace{u(x,0) = u_0(x)\,,\quad \frac{\partial u}{\partial t}(x,0) = v_0(x)}_{\text{initial}}\,,\quad x\in\mathbb{R}\,. \tag{6.3.2.2}$$

Trick: Use characteristic coordinates

$$\xi = x + ct\;,\quad \tau = x - ct:\quad \tilde{u}(\xi,\tau) := u\left(\frac{\xi+\tau}{2}, \frac{\xi-\tau}{2c}\right)\,. \tag{6.3.2.3}$$

chain rule

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \;\blacktriangleright\; \frac{\partial^2 \tilde{u}}{\partial\xi\partial\tau} = 0 \;\Rightarrow\; \tilde{u}(\xi,\tau) = F(\xi) + G(\tau)\,,$$

for **any** $F, G \in C^2(\mathbb{R})$!

$$u(x,t) = F(x+ct) + G(x-ct)$$

F, G from initial conditions

$$u_0(x) = u(x,0) = F(x) + G(x)$$
$$v_0(x) = \frac{\partial u}{\partial t}(x,0) = c\,(F'(x) - G'(x)) \quad \Big| \int_{-\infty}^{x}$$

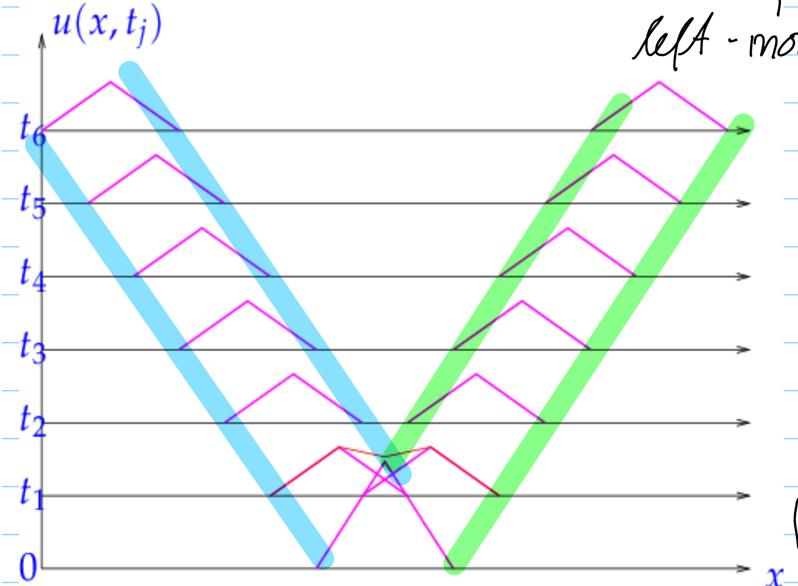
compactly supported

▶ $$\boxed{u(x,t) = \tfrac{1}{2}(u_0(x+ct) + u_0(x-ct)) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,\mathrm{d}s} \,. \tag{6.3.2.4}$$

[d'Alembert solution]

Special case: $v_0 \equiv 0$, u_0 compactly supported

$$u(x,t) = \tfrac{1}{2}(\underbrace{u_0(x+ct)}_{\text{left-moving}} + \underbrace{u_0(x-ct)}_{\text{right-moving}})$$



with speed $c > 0$

$\Downarrow$

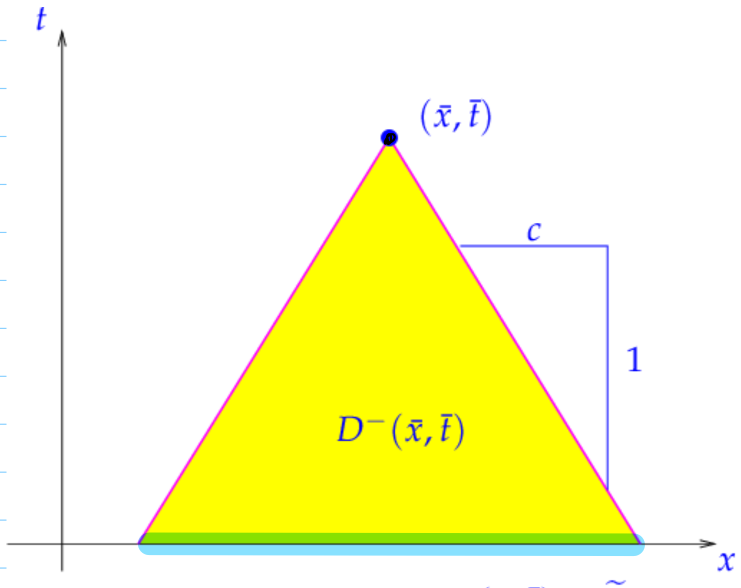
finite speed c of propagation

(compact supports preserved)

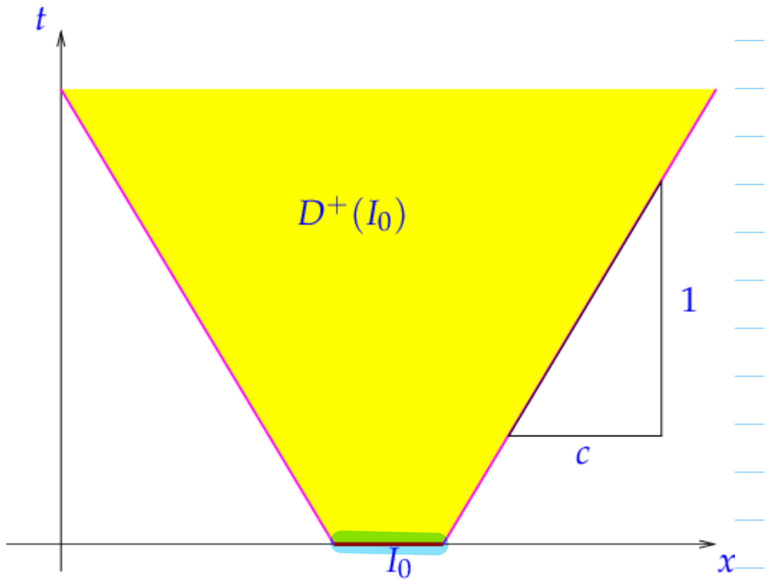
Remark: For 1D heat equation, Cauchy problem

Even if $\operatorname{supp} u_0$ bounded $\Rightarrow$ $\operatorname{supp} u(\cdot,t) = \mathbb{R}$ $\forall t > 0$

Another consequence of finite speed of propagation



domain of dependence of $(\bar{x},\bar{t}) \in \tilde{\Omega}$



domain of influence of $I_0 \subset \mathbb{R}$

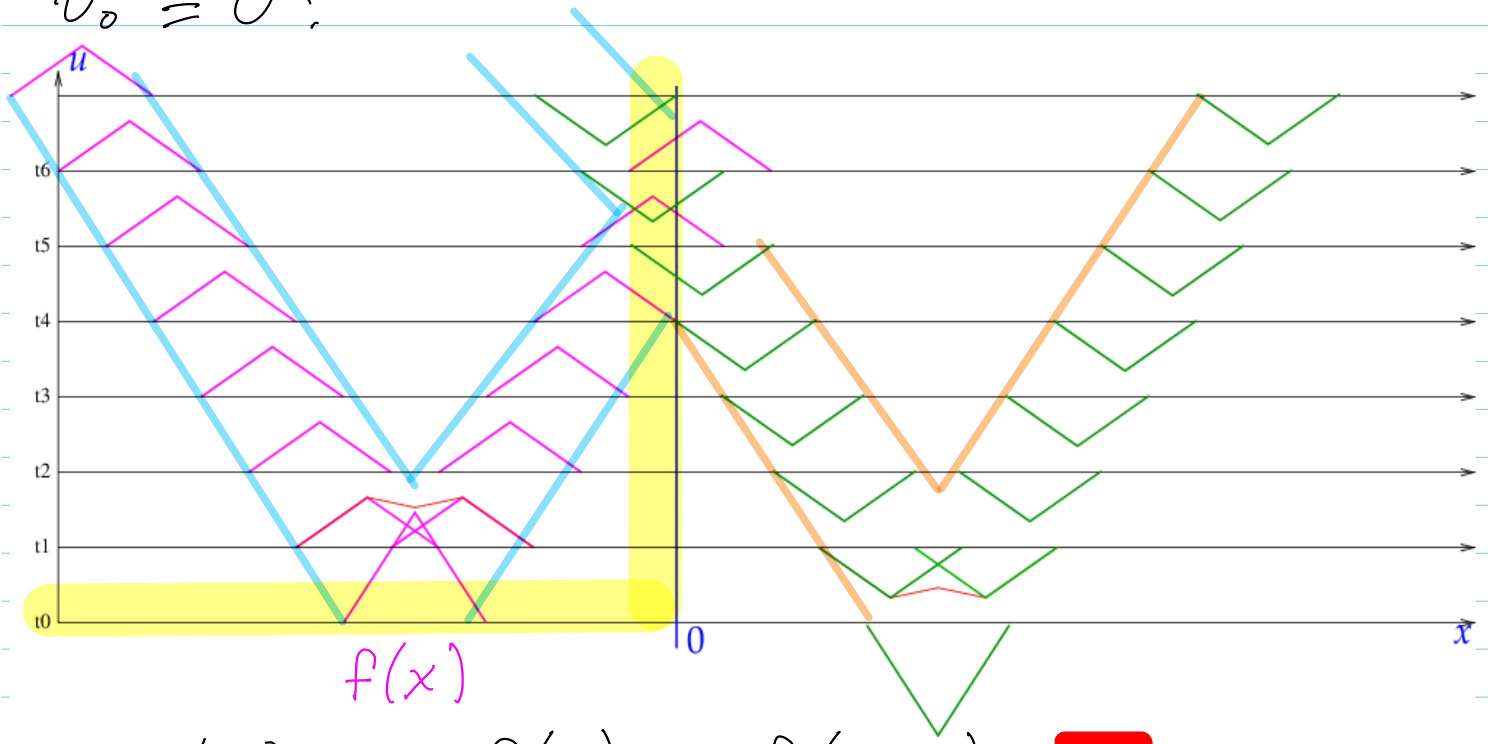
$u(\bar{x},\bar{t})$ depends only on $u(x,t)$ in

$u_{|I_0}, \frac{\partial u}{\partial t}_{|I_0}$ impact solution only in

▷ Also carries over to higher dimensions

③ Remark: Spatial boundary condition

$v_0 \equiv 0$:



$u_0(x) = f(x) - f(-x)$

$\triangleright$ [d'Alembert]: $u(0,t) = 0$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{on} \quad \mathbb{R}_- \times [0,T]\,,$$
$$u(x,0) = u_0(x)\,, \quad \frac{\partial u}{\partial t}(x,0) = 0\,, \quad x \leq 0\,, \tag{6.3.2.11}$$
$$u(0,t) = 0 \quad \text{for} \quad t \geq 0 \quad \text{(Dirichlet b.c.)}\,.$$

Reflection at boundaries

Homogeneous Dirichlet or Neumann boundary conditions on (parts of) $\partial\Omega$ model the reflection of waves (at those parts of $\partial\Omega$).

Conservation of total energy (w/o forcing)

$$u : [0,T] \to H_0^1(\Omega): \quad \int_\Omega \rho(x) \cdot \frac{\partial^2 u}{\partial t^2} v \,\mathrm{d}x + \int_\Omega \sigma(x)\, \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x = 0 \quad \forall v \in H_0^1(\Omega) \tag{6.3.2.14}$$

$$\updownarrow$$

$$\boxed{u \in V_0: \quad \mathsf{m}(\ddot{u}, v) + \mathsf{a}(u,v) = 0 \quad \forall v \in V_0}\,, \tag{6.3.2.15}$$

Total energy $E(t) = \underbrace{\tfrac{1}{2} a(u(t), u(t))}_{\text{elastic energy / potential energy}} + \underbrace{\tfrac{1}{2} m(\dot{u}(t), \dot{u}(t))}_{\text{kinetic energy}}$

Theorem 6.3.2.16. Energy conservation in wave propagation

If $u : \tilde{\Omega} \mapsto \mathbb{R}$ solves (6.3.2.15), then we have conservation of total energy *in the sense that*

$$t \mapsto \tfrac{1}{2}\mathsf{m}\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right) + \tfrac{1}{2}\mathsf{a}(u,u) \equiv \text{const}\,.$$

kinetic energy *elastic (potential) energy*, *see* (1.2.1.19)

Proof: general product for symmetric bilinear forms

$$\frac{d}{dt} E(t) \stackrel{\downarrow}{=} a(u(t), \dot{u}(t)) + m(\dot{u}(t), \ddot{u}(t)) = 0$$

$= v$ in (6.3.2.15)

Review questions 9.3.2.17

[Please answer without relying on notes]

A :

We consider the Cauchy problem for the constant-coefficient linear wave equation in $\mathbb{R}^d$:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^d \times \mathbb{R}^+ . \tag{6.3.2.18}$$

For $\boldsymbol{k} \in \mathbb{R}^d$, $\|\boldsymbol{k}\| = 1$, a solution of (6.3.2.18) of the form

$$u(\boldsymbol{x}, t) = \operatorname{Re} \exp(i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)) , \quad \omega \in \mathbb{R} , \tag{6.3.2.19}$$

is called a plane-wave solution. Find ω as a function of $\boldsymbol{k}$.

B :

Consider the Cauchy problem for the 1D wave equation

$$c > 0: \ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad u(x,0) = u_0(x) , \quad \frac{\partial u}{\partial t}(x,0) = v_0(x) , \quad x \in \mathbb{R} , \tag{6.3.2.2}$$

and recall the d'Alembert solution

$$u(x,t) = \tfrac{1}{2}(u_0(x+ct) + u_0(x-ct)) + \tfrac{1}{2c} \int_{x-ct}^{x+ct} v_0(s) \, \mathrm{d}s . \tag{6.3.2.4}$$

Which relationship must be satisfied for the initial data u_0 and v_0 such that you obtain a solution that propagates only to the right?

C :

We study the following initial-boundary value problem for the 1D linear constant-coefficeint wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{in} \quad]0,1[\times \mathbb{R}^+ , \\ u(0,t) = u(1,t) &= 0 \quad \forall t > 0 , \\ u(x,0) = u_0(x) \quad , \quad \frac{\partial u}{\partial t}(x,0) &= v_0(x) , \quad 0 < x < 1 . \end{aligned} \tag{6.3.2.20}$$

1. In the $x - t$-plane sketch the precice domain of influence of the interval $[\frac{1}{4}, \frac{3}{4}]$.
2. What is the domain of dependence for the point $\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$?

Hint. Remember that homogeneous Dirichlet boundary conditions for the 1D wave equation model a reflection of the wave at the boundary.

D :

Consider the abstract variational wave equation

$$u = u(t) \in V_0 : \mathsf{m}(\ddot{u}, v) + +\mathsf{b}(\dot{u}, v) + \mathsf{a}(u, v) = 0 \quad \forall v \in V_0 , \tag{6.3.2.21}$$

where V_0 is a vector space and m, a are symmetric positive definite bilinear forms on V_0. Which properties of the bilinear form b ensure an *exponential decay* of the energy

$$\mathcal{E}(t) := \tfrac{1}{2}\mathsf{m}(\dot{u}(t), \dot{u}(t)) + \tfrac{1}{2}\mathsf{a}(u(t), u(t))$$

when $t \mapsto u(t)$ solves (6.3.2.21).

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