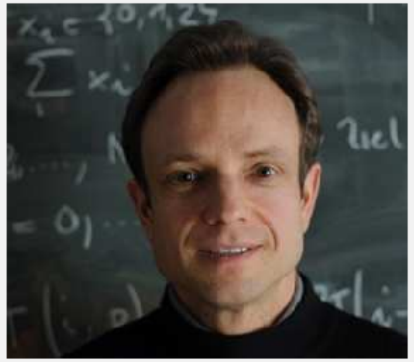
①

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 9.3.3: Method of Lines for Wave Propagation



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Dependency. [Lecture → Section 9.3.2], [Lecture → **??**], and [Lecture → Section 2.6]. Useful is [Lecture → Section 9.2.4].

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

Duration: 13 minutes

VI. Second-Order Linear Evolution Problems

9.3 Linear Wave Equations

9.3.3. Method of Lines for Wave Propagation

▷ Same idea as for heat equation
Galerkin in space

test space does not depend on time

$$t\in]0,T[\mapsto u(t)\in V_0 \;:\; \begin{cases} \mathsf{m}\left(\dfrac{d^2u}{dt^2}(t),v\right)+\mathsf{a}(u(t),v) = \underbrace{0}_{\ell(t)(v)} \quad \forall v\in V_0\,, \\ u(0)=u_0\in V_0 \;,\quad \dfrac{du}{dt}(0)=v_0\in V_0\,. \end{cases} \tag{6.3.3.2}$$

[initial condition]

I : Replace $V_0 \rightarrow V_{0,h}\subset V_0$, $\dim V_{0,h} =: N$

independent of time

$$t\in]0,T[\mapsto u(t)\in V_{0,h} \;:\; \begin{cases} \mathsf{m}\left(\dfrac{d^2u_h}{dt^2}(t),v_h\right)+\mathsf{a}(u_h(t),v_h) = \cancel{0}\;\ell(t)(v_h) \quad \forall v_h\in V_{0,h}\,, \\ u_h(0) = \text{projection/interpolant of } u_0 \text{ in } V_{0,h}\,, \\ \dfrac{du_h}{dt}(0) = \text{projection/interpolant of } v_0 \text{ in } V_{0,h}\,. \end{cases} \tag{6.3.3.3}$$

approximate initial cond.

2

II. Ordered basis $\mathfrak{B} = \{b_h^1, \dots, b_h^N\} \subset V_{0,h}$

$$u_h(t) = \sum_{j=1}^{N} \mu_j(t)\, b_h^j$$

▷ LWE - MOL - ODE : (2nd-order)

(6.3.3.3) $\Rightarrow$
$$\begin{cases} \mathbf{M}\left\{\frac{d^2}{dt^2}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) & \text{for } 0 < t < T\,, \\ \vec{\mu}(0) = \vec{\mu}_0 \;,\; \frac{d\vec{\mu}}{dt}(0) = \vec{v}_0\,. \end{cases} \tag{6.3.3.4}$$

↘ state space $\mathbb{R}^N$

▷ s.p.d. stiffness matrix $\mathbf{A} \in \mathbb{R}^{N,N}$, $(\mathbf{A})_{ij} := \mathsf{a}(b_h^j, b_h^i)$ (independent of time),
▷ s.p.d. mass matrix $\mathbf{M} \in \mathbb{R}^{N,N}$, $(\mathbf{M})_{ij} := \mathsf{m}(b_h^j, b_h^i)$ (independent of time),
▷ source (load) vector $\vec{\varphi}(t) \in \mathbb{R}^N$, $(\vec{\varphi}(t))_i := \ell(t)(b_h^i)$ (time-dependent),
▷ $\vec{\mu}_0 \hat{=}$ coefficient vector of a projection of u_0 onto $V_{0,h}$.
▷ $\vec{v}_0 \hat{=}$ coefficient vector of a projection of v_0 onto $V_{0,h}$.

▷ $E_h(t) = \frac{1}{2}\vec{\mu}(t)^T \mathbf{A}\vec{\mu}(t) + \frac{1}{2}\dot{\vec{\mu}}(t)^T \mathbf{M}\dot{\vec{\mu}}(t)$

↳ conserved discrete energy

Conversion to 1-st-order ODE :

Auxiliary function : $\vec{v}(t) = \dot{\vec{\mu}}(t)$

2nd-order ODE : $\mathbf{M}\left\{\frac{d^2}{dt^2}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = 0$

$\leftarrow$ auxiliary unknown $\vec{v} = \dot{\vec{\mu}}$

State space $\mathbb{R}^{2N}$

$$\begin{cases} \frac{d}{dt}\vec{\mu}(t) = \vec{v}(t)\,, \\ \mathbf{M}\frac{d}{dt}\vec{v}(t) = -\mathbf{A}\vec{\mu}(t)\,, \end{cases} \;,\quad 0 < t < T\,. \tag{6.3.3.6}$$

with intial conditions

$$\vec{\mu}(0) = \vec{\mu}_0 \;,\quad \vec{v}(0) = \vec{v}_0\,. \tag{6.3.3.7}$$

3

Review question 9.3.3.9 :

[to be solved without looking at notes]

A :

The method-of-lines semi-discretization of the variatonal evolution equation

$$t \in [0,T] \mapsto u(t) \in V_0: \quad \mathsf{m}\left(\frac{\partial^2 u}{\partial t^2}(t), v\right) + \mathsf{a}(u(t), v) = \ell(t)(v) \quad \forall v \in V_0 ,$$

V_0 a vector space, leads to an ordinary differential equation of the form

$$\mathbf{M}\frac{d^2\vec{\boldsymbol{\mu}}}{dt^2}(t) + \mathbf{A}\vec{\boldsymbol{\mu}}(t) = \vec{\boldsymbol{\varphi}}(t) ,$$

with $t \mapsto \vec{\boldsymbol{\mu}}(t) \in \mathbb{R}^N$, $t \mapsto \vec{\boldsymbol{\varphi}}(t) \in \mathbb{R}^N$, and matrices $\mathbf{A}, \mathbf{M} \in \mathbb{R}^{N,N}$.

1. Give formulas for the entries of $\mathbf{A}$, $\mathbf{M}$, and $\vec{\boldsymbol{\varphi}}(t)$.
2. What is the meaning of $t \mapsto \vec{\boldsymbol{\mu}}(t)$?

B :

In order to convert a variational evolution problem into an ordinary differential equation following the policy of the method of lines, we have to choose a basis of the discrete trial and test space $V_{0,h} \subset V_0$. How does the solution $t \mapsto u_h(t) \in V_{0,h}$ of the semi-discrete evolution problem depend on the choice of basis?

C :

We consider the hyperbolic linear evolution problem

$$u(t) \in H_0^1(\Omega): \quad \int_\Omega \rho(\boldsymbol{x},t)\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t)v(\boldsymbol{x}) + \sigma(\boldsymbol{x},t)\,\mathbf{grad}\,u(\boldsymbol{x}) \cdot \mathbf{grad}\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall v \in H_0^1(\Omega) ,$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad , \quad \frac{\partial u}{\partial t}(\boldsymbol{x},0) = v_0(\boldsymbol{x}) , \quad \boldsymbol{x} \in \Omega ,$$

where $\rho : [0,T] \times \Omega \to \mathbb{R}$ and $\sigma : [0,T] \times \Omega \to \mathbb{R}$ are uniformly positive coefficient functions.
We perform a method of lines spatial semi-discretization based on $V_{0,h} \subset H_0^1(\Omega)$ equipped with a basis $\{b_h^1, \dots, b_h^N\}$, $N := \dim V_{0,h}$.

1. Write down the resulting ordinary differential equation (ODE) and characterize its building blocks.
2. Describe how one can obtain the initial values for the method-of-lines ODE, if $V_{0,h}$ is a Lagrangian finite element space.