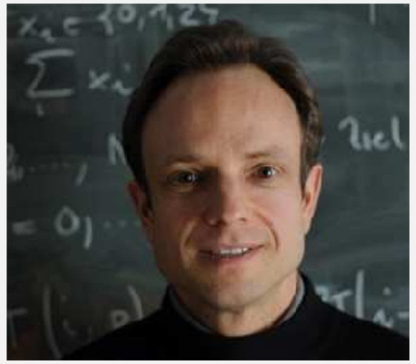
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 9.3.4: Timestepping for Semi-Discrete Wave Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Date: April 28, 2021



Dependency. [Lecture → Section 9.3.3] and [Lecture → Section 9.2.6]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

Duration : 43 minutes

VI. Second-Order Linear Evolution Problems

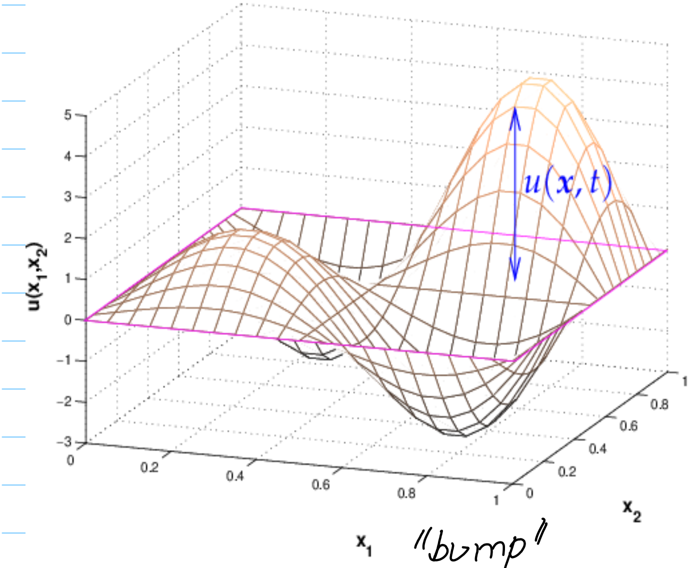
9.3 Linear Wave Equations

9.3.4. Timestepping for Semi-Discrete Wave Equations

$$(6.3.3.3) \quad \Rightarrow \quad \begin{cases} \mathbf{M}\left\{\frac{d^2}{dt^2}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \text{ for } 0 < t < T\,, \\ \vec{\mu}(0) = \vec{\mu}_0 \quad , \quad \frac{d\vec{\mu}}{dt}(0) = \vec{\nu}_0\,. \end{cases} \qquad (9.3.3.4)$$

Structural properties : - Energy conservation
- Reversibility $(t \Leftrightarrow -t)$

② Experiment :



Model problem: wave propagation on a square membrane

no forcing

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{on} \quad]0,1[^2 \times]0,1[\,,$$
$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times]0,T[\,,$$
$$u(x,0) = u_0(x) \quad , \quad \frac{\partial u}{\partial t}(x,0) = 0 \,. \qquad v_0 \equiv 0$$

(membrane in flat frame, displaced, but at rest at initial time)

- ✦ Initial data $u_0(x) = \max\{0, \frac{1}{5} - \|x\|\}$, $v_0(x) = 0$, → $n = 30$
- ✦ $\mathcal{M} \mathrel{\hat=}$ "structured triangular tensor product mesh", see Fig. 232, n squares in each direction,
- ✦ linear finite element space $V_{N,0} = \mathcal{S}^0_{1,0}(\mathcal{M})$, $N := \dim \mathcal{S}^0_{1,0}(\mathcal{M}) = (n-1)^2$,
- ✦ All local computations (→ Section 2.7.5) rely on 3-point vertex based local quadrature formula "2D trapezoidal rule" (2.4.6.10). More explanations will be given in Rem. 6.3.4.18 below.

- Expl./Impl. Euler SSM : $\dot{u} = f(u)$

$$\Rightarrow u^{(j+1)} = u^{(j)} + \tau f(u^{(j+1)})$$

(6.3.3.3) $\Rightarrow$
$$\begin{cases} \mathbf{M}\left\{\frac{d^2}{dt^2}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = 0 & \text{for } 0 < t < T \,, \\ \vec{\mu}(0) = \vec{\mu}_0 \quad , \quad \frac{d\vec{\mu}}{dt}(0) = \vec{\nu}_0 \,. \end{cases} \qquad (9.3.3.4)$$

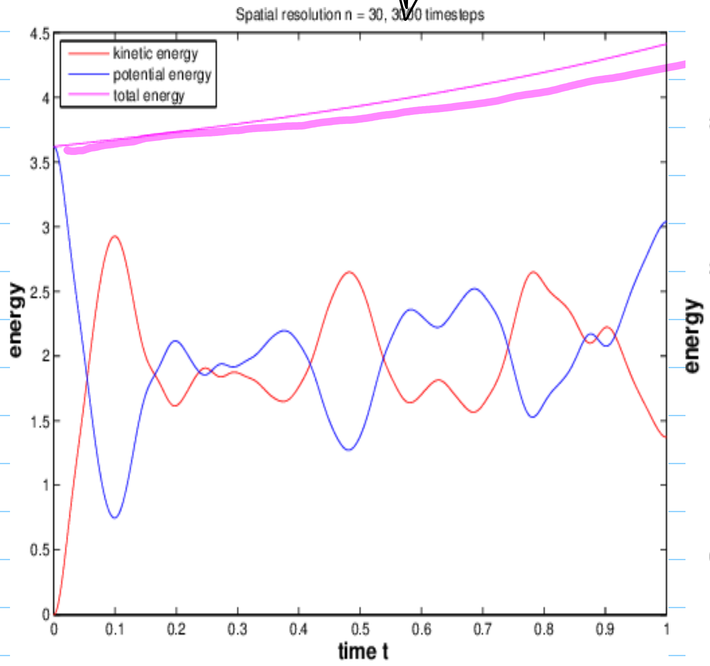
Aux. var. $\vec{\nu} = \dot{\vec{\mu}}(t)$: $\quad \frac{d}{dt}\begin{bmatrix} \vec{\mu} \\ \vec{\nu} \end{bmatrix} = \begin{bmatrix} \vec{\nu} \\ -M^{-1}A\vec{\mu} \end{bmatrix}$

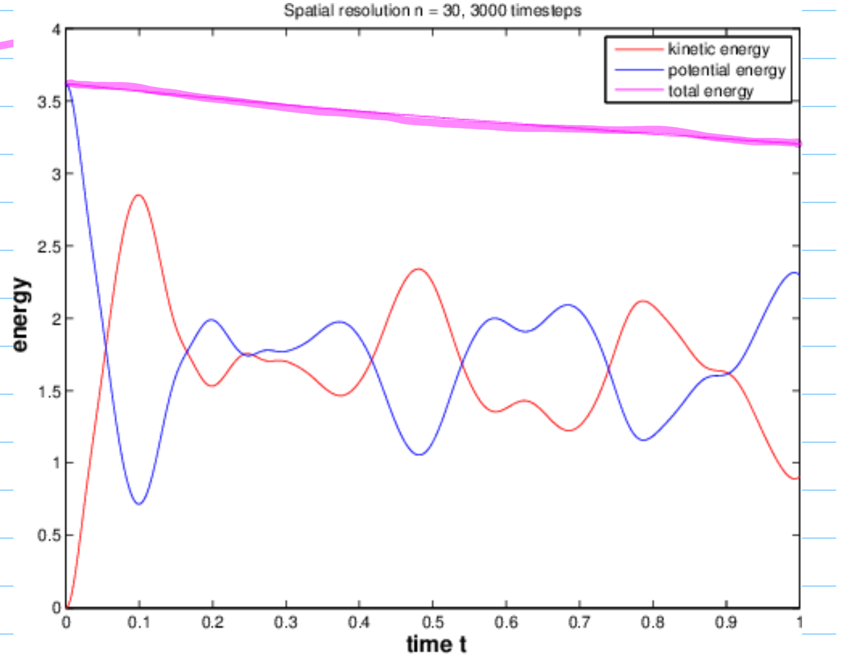
$[\,\vec{\varphi} = 0\,]$

Expl. Euler : [Impl.]
$$\frac{\mu^{(j)} - \mu^{(j-1)}}{\tau} = \nu^{(j-1)}$$
$$\frac{\vec{\nu}^{(j)} - \nu^{(j-1)}}{\tau} = -M^{-1}A\vec{\mu}^{(j-1)}$$

explicit Euler	implicit Euler
$\vec{\mu}^{(j)} - \vec{\mu}^{(j-1)} = \tau\vec{\nu}^{(j-1)}$, $\quad \mathbf{M}(\vec{\nu}^{(j)} - \vec{\nu}^{(j-1)}) = -\tau\mathbf{A}\vec{\mu}^{(j-1)}$.	$\vec{\mu}^{(j)} - \vec{\mu}^{(j-1)} = \tau\vec{\nu}^{(j)}$, $\quad \mathbf{M}(\vec{\nu}^{(j)} - \vec{\nu}^{(j-1)}) = -\tau\mathbf{A}\vec{\mu}^{(j)}$.

Observed : blow-up decay





exp increase of energy $\qquad$ exp. decay of energy

③

Exp.: Implicit midpoint SSM $\hat{=}$ 1-stage Gauss collocation RK-SSM

$$\Psi^{t,t+\tau}\mathbf{u} := \mathbf{w}: \quad \mathbf{w} = \mathbf{u} + \tau\mathbf{f}(t+\tfrac{1}{2}\tau, \tfrac{1}{2}(\mathbf{w}+\mathbf{u}))\,, \qquad (9.2.6.7)$$

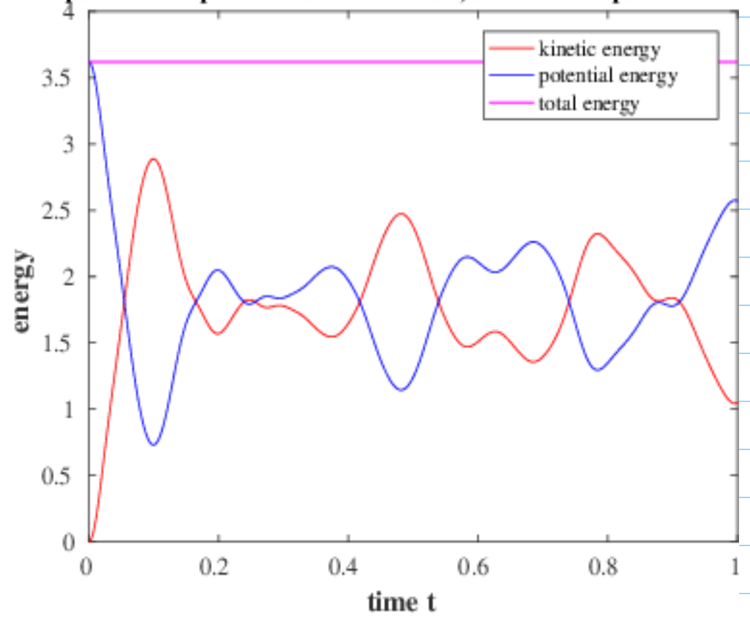
$$\frac{w-u}{\tau} = f(\tfrac{1}{2}(w+u))$$

▷ For LWE-MOL-ODE

$$\frac{\mu^{(j)}-\mu^{(j-1)}}{\tau} = \tfrac{1}{2}(\vec{v}^{(j)}+\vec{v}^{(j-1)})$$

$$\frac{\vec{v}^{(j)}-v^{(j-1)}}{\tau} = -\tfrac{1}{2}M^{-1}A(\vec{\mu}^{(j)}+\mu^{(j-1)})$$

Implicit midpoint rule: spatial resolution n = 30, 3000 timesteps



◁ Exact conservation of energy !

Rigorous analysis: (timestep τ)

$$\begin{bmatrix}\vec{\mu}^{(j)}\\ \vec{v}^{(j)}\end{bmatrix} = \begin{bmatrix}\vec{\mu}^{(j-1)}\\ \vec{v}^{(j-1)}\end{bmatrix} + \tau\begin{bmatrix}\frac{1}{2}(\vec{v}^{(j-1)}+\vec{v}^{(j)})\\ -\frac{1}{2}\mathbf{M}^{-1}\mathbf{A}(\vec{\mu}^{(j-1)}+\vec{\mu}^{(j)})\end{bmatrix} + \tau\begin{bmatrix}0\\ \mathbf{M}^{-1}\vec{\varphi}(t_j-\frac{1}{2}\tau)\end{bmatrix}.$$

$$\Updownarrow$$

$$\begin{bmatrix}\mathbf{I} & -\frac{1}{2}\tau\mathbf{I}\\ \frac{1}{2}\tau\mathbf{A} & \mathbf{M}\end{bmatrix}\begin{bmatrix}\vec{\mu}^{(j)}\\ \vec{v}^{(j)}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \frac{1}{2}\tau\mathbf{I}\\ -\frac{1}{2}\tau\mathbf{A} & \mathbf{M}\end{bmatrix}\begin{bmatrix}\vec{\mu}^{(j-1)}\\ \vec{v}^{(j-1)}\end{bmatrix} + \tau\begin{bmatrix}0\\ \vec{\varphi}(t_j-\frac{1}{2}\tau)\end{bmatrix},$$

$$\Updownarrow$$

$$(\vec{\mu}^{(j-1)}+\vec{\mu}^{(j)})^T A^T \cdot \Big| \quad \vec{\mu}^{(j)}-\vec{\mu}^{(j-1)} = \tfrac{1}{2}\tau(\vec{v}^{(j-1)}+\vec{v}^{(j)}) \qquad (*)$$

$$(\vec{v}^{(j-1)}+\vec{v}^{(j)})^T \cdot \Big| \quad \mathbf{M}(\vec{v}^{(j)}-\vec{v}^{(j-1)}) = -\tfrac{1}{2}\tau\mathbf{A}(\vec{\mu}^{(j-1)}+\vec{\mu}^{(j)}) + \tau\vec{\varphi}(t_j-\tfrac{1}{2}\tau).$$

Use "$a^2-b^2=(a+b)(a-b)$": $(\vec{\alpha}+\vec{\beta})^T A(\vec{\alpha}-\vec{\beta}) = \vec{\alpha}^T A\vec{\alpha} - \vec{\beta}^T A\vec{\beta}$ for $A = A^T$

Add equ.:

$$(\vec{\mu}^{(j)})^\top\mathbf{A}(\vec{\mu}^{(j)}) - (\vec{\mu}^{(j-1)})^\top\mathbf{A}(\vec{\mu}^{(j-1)}) + (\vec{v}^{(j)})^\top\mathbf{M}(\vec{v}^{(j)}) - (\vec{v}^{(j-1)})^\top\mathbf{M}(\vec{v}^{(j-1)})$$

$$= \cancel{\tfrac{1}{2}\tau\Big((\vec{\mu}^{(j)}+\vec{\mu}^{(j-1)})^\top\mathbf{A}(\vec{v}^{(j)}+\vec{v}^{(j-1)}) - (\vec{v}^{(j)}+\vec{v}^{(j-1)})^\top\mathbf{A}(\vec{\mu}^{(j)}+\vec{\mu}^{(j-1)})\Big)} + \quad \text{since } = 0,\ A = A^T$$

$$\tau(\vec{\mu}^{(j)}+\vec{\mu}^{(j-1)})^\top\vec{\varphi}(t_j-\tfrac{1}{2}\tau)$$

$$= \tau(\vec{\mu}^{(j)}+\vec{\mu}^{(j-1)})^\top\vec{\varphi}(t_j-\tfrac{1}{2}\tau), \Leftarrow 0, \text{ if } \vec{\varphi} \equiv 0$$

$$\hookrightarrow = 2(E_h^{(j)} - E^{(j-1)})$$

(4) Remark: Elimination of $\vec{v}$ from (*)

[consider (*) for j & $j+1$ and subtract]

$$\mathbf{M}\left\{\frac{d^2}{dt^2}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \qquad (9.3.4.2)$$

$$\Downarrow$$

$$\boxed{\mathbf{M}\frac{\vec{\mu}^{(j+1)} - 2\vec{\mu}^{(j)} + \vec{\mu}^{(j-1)}}{\tau^2} = -\frac{1}{4}\mathbf{A}(\vec{\mu}^{(j-1)} + 2\vec{\mu}^{(j)} + \vec{\mu}^{(j+1)}) + \frac{1}{2}(\vec{\varphi}(t_j - \tfrac{1}{2}\tau) + \vec{\varphi}(t_j + \tfrac{1}{2}\tau)),\quad j = 1,2,\dots} \qquad (9.3.4.12)$$

2nd difference quotient

Obviously reversible

≙ Crank-Nicolson timestepping

A 2-step method → need special starting step

$$\frac{d}{dt}\vec{\mu}(0) = \vec{v}_0 \quad \blacktriangleright \quad \frac{\vec{\mu}^{(1)} - \vec{\mu}^{(-1)}}{2\tau} = \vec{v}_0 . \qquad (9.3.4.15)$$

A simplification: Evaluate ___ at j-th timestep

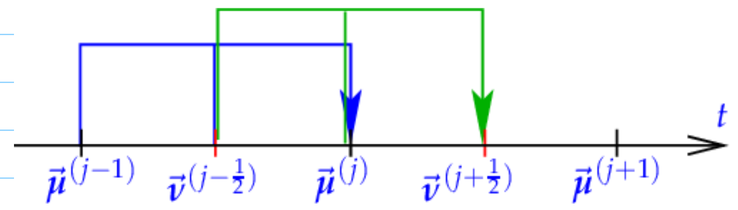
$\vec{\varphi} \equiv 0$: $\Rightarrow \quad M\dfrac{\vec{\mu}^{(j+1)} - 2\mu^{(j)} - \mu^{(j-1)}}{\tau^2} = -A\vec{\mu}^{(j)} \quad \Leftrightarrow$

For $\vec{\varphi} \equiv 0$:

$$\boxed{\begin{aligned} \mathbf{M}\frac{\vec{v}^{(j+\frac{1}{2})} - \vec{v}^{(j-\frac{1}{2})}}{\tau} &= -\mathbf{A}\vec{\mu}^{(j)} , \\ \frac{\vec{\mu}^{(j+1)} - \vec{\mu}^{(j)}}{\tau} &= \vec{v}^{(j+\frac{1}{2})} , \end{aligned} \quad j = 0,1,\dots} , \qquad (9.3.4.17)$$

$+$ initial step $\vec{v}^{(-\frac{1}{2})} + \vec{v}^{(\frac{1}{2})} = 2\vec{v}_0$.

Leapfrog implementation: in turns evaluate both equations



work per step:
1× evaluation $\mathbf{A}$ × vector,
1× solution of linear system for $\mathbf{M}$

Can be simplified by mass lumping:

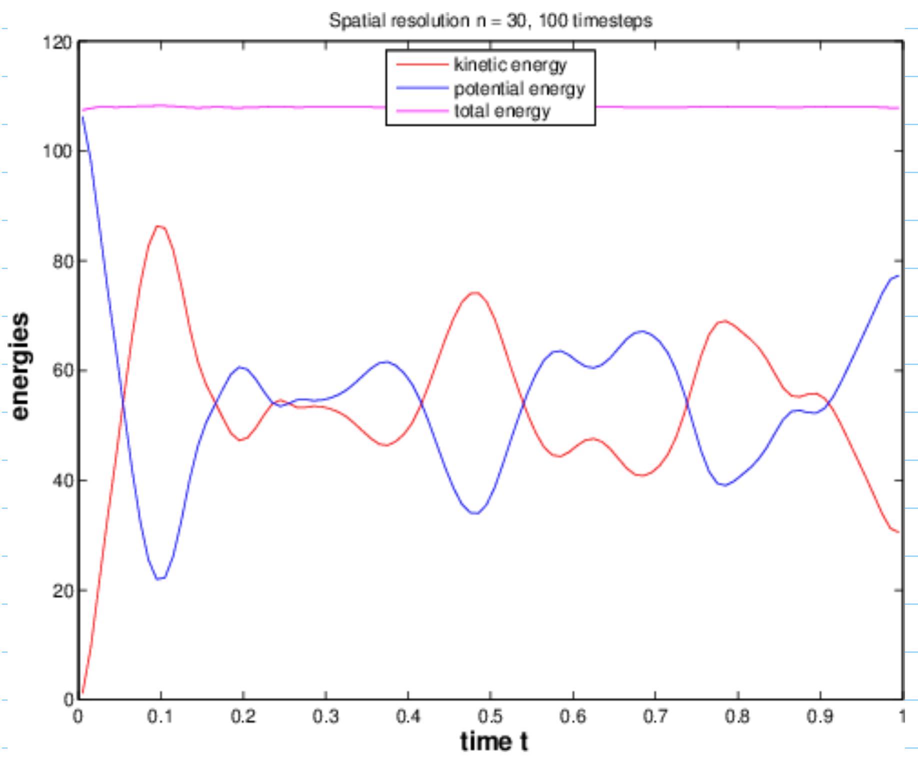
↳ "Make M diagonal"

For $S_1^0(\mathcal{M})$: use vertex-based local quadrature

→ M diagonal because GSFs are a cardinal basis w.r.t. nodes of the mesh

LF becomes explicit
[much cheaper than CR]

5 ☹ LF does not conserve total energy, BUT



◁ Energy "almost" conserved.

no drift of energy.

[LF is a symplectic timestepping scheme]

6

Review questions 9.3.4.21

[try to answer without consulting notes.]

A:

Remember that Crank-Nicolson timestepping for the linear ODE

$$\mathbf{M}\frac{d^2\vec{\mu}}{dt^2}(t) + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t)$$

gives rise to the recursion

$$\mathbf{M}\,\frac{\vec{\mu}^{(j+1)} - 2\vec{\mu}^{(j)} + \vec{\mu}^{(j-1)}}{\tau^2} = -\tfrac{1}{2}\mathbf{A}(\vec{\mu}^{(j-1)} + \vec{\mu}^{(j+1)}) + \tfrac{1}{2}(\vec{\varphi}(t_j - \tfrac{1}{2}\tau) + \vec{\varphi}(t_j + \tfrac{1}{2}\tau))\,, \quad j = 1,2,\ldots\,,$$

while the Störmer-Verlet timestepping yields

$$\mathbf{M}\,\frac{\vec{\mu}^{(j+1)} - 2\vec{\mu}^{(j)} + \vec{\mu}^{(j-1)}}{\tau^2} = -\mathbf{A}\vec{\mu}^{(j)} + \vec{\varphi}(t_j)\,.$$

Generalize these timestepping schemes to the second-order ODE

$$\ddot{\mathbf{u}}(t) = \mathbf{f}(t,\mathbf{u}(t)) \quad , \quad \mathbf{f} : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n\,, \quad j = 1,2,3,\ldots\,.$$

B:

We consider the linear non-autonomous ODE

$$\frac{d\vec{\mu}}{dt}(t) = \mathbf{A}(t)\vec{\mu}(t) \quad [\,\vec{\mu}(t) \in \mathbb{R}^N\,]\,, \tag{6.3.4.22}$$

with a *skew-symmetric* matrix-valued function $t \mapsto \mathbf{A}(t) \in \mathbb{R}^{N,N}$, that is, $\mathbf{A}(t)^\top = -\mathbf{A}(t)$. We apply the implicit midpoint method with uniform timestep $\tau > 0$ to discretize (6.3.4.22) in time, which produces the sequence $\vec{\mu}^{(j)}$, $j = 0,1,2,\ldots$. Show that $\left\|\vec{\mu}^{(j)}\right\| = \left\|\vec{\mu}^{(0)}\right\|$ for all j.

Hint. The discrete evolution operator of the implicit midpoint rule when applied to the ODE $\dot{\mathbf{u}} = \mathbf{f}(t,\mathbf{u})$ is

$$\mathbf{\Psi}^{t,t+\tau}\mathbf{u} := \mathbf{w}: \quad \mathbf{w} = \mathbf{u} + \tau\mathbf{f}(t + \tfrac{1}{2}\tau, \tfrac{1}{2}(\mathbf{w}+\mathbf{u}))\,.$$

C:

What is mass laumping and why is it important in the context of the leapfrog timestepping scheme

$$\begin{aligned}\mathbf{M}\,\frac{\vec{\nu}^{(j+\frac{1}{2})} - \vec{\nu}^{(j-\frac{1}{2})}}{\tau} &= -\mathbf{A}\vec{\mu}^{(j)}\,,\\ \frac{\vec{\mu}^{(j+1)} - \vec{\mu}^{(j)}}{\tau} &= \vec{\nu}^{(j+\frac{1}{2})}\,,\end{aligned} \qquad j = 0,1,\ldots$$

for the method-of-lines ODE

$$\mathbf{M}\left\{\frac{d^2}{dt^2}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = 0\ ?$$

D:

Explain the statement

> There is *no drift* of total energy when using leapfrog timestepping for the method-of-lines ODE for a linear wave equation.