ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2020

(C) Seminar für Angewandte Mathematik, ETH Zürich

Q & A Session
May 4, 2020

HW Problem 2-15: (Regularized pure Neumann Problem)

$u \in H^1(\Omega)$

$$\int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,dx = \underbrace{\int_\Omega f v \,dx + \int_{\partial\Omega} h v \,dS} \quad \forall v \in H^1(\Omega)$$

$\downarrow$

No unique solutions : Must satisfy compatibility condition

c) FE-Galerkin discretization based on

$$S^0_{1,*}(\mathcal{M}) := \{ v_h \in S^0_1(\mathcal{M}) : v_h(p) = 0 \}$$

$p \mathrel{\hat=}$ node of $\mathcal{M}$ w/ index $= 1$

Start from full $S^0_1(\mathcal{M})$ Galerkin linear system

$$\begin{bmatrix} a_{11} & \mathbf{a}_1^T \\ \mathbf{a}_1 & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vec{\mu} \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \vec{\varphi} \end{bmatrix} \mathrel{\hat=} N \times N\text{-LSE} \quad (*)$$

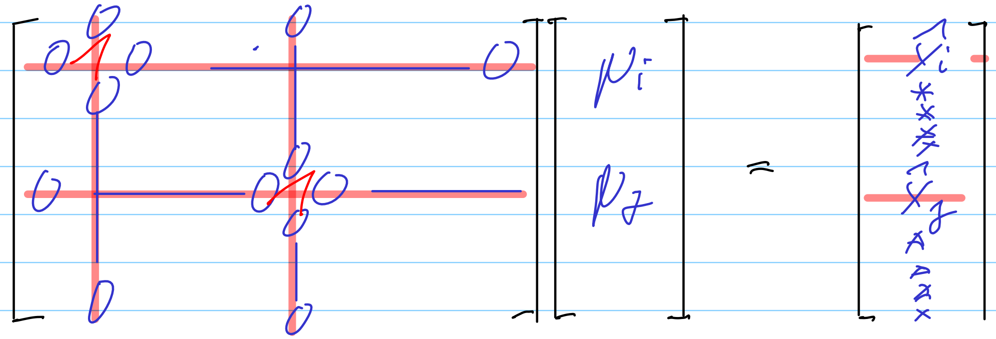
$N := \dim S^0_1(\mathcal{M})$

$S^0_1(\mathcal{M}) \longrightarrow S^0_{1,*}(\mathcal{M}) \mathrel{\hat=}$ drop b_h^1

$\mathrel{\hat=}$ drop 1st row & col. of $(*)$

② In LF++ :

→ Fix Flagged Solution Components*



* can be used to remove basis functions

2.15 d) – h) :

$u \in H^1(\Omega)\,[\to S_1^0(\mathcal{M})],\ \lambda \in \mathbb{R}$

$$\left\{\begin{aligned} \int_\Omega \mathbf{grad}\,u \cdot \mathbf{grad}\,v\,\mathrm{d}x + \lambda \int_\Omega v\,\mathrm{d}x &= \int_\Omega f v\,\mathrm{d}x + \int_{\partial\Omega} h v\,\mathrm{d}S \quad \forall v \in H^1(\Omega) \\ \int_\Omega u\,\mathrm{d}x &= 0\,. \end{aligned}\right. \tag{2.15.8}$$

$\Leftrightarrow S_1^0(\mathcal{M})$

$\hat{=}$ a linear variational problem

$$\begin{array}{c} N \{ \\ 1 \{ \end{array} \begin{bmatrix} A & \underline{c} \\ \underline{c}^T & 0 \end{bmatrix} \begin{bmatrix} \vec{\mu} \\ \lambda \end{bmatrix} = \begin{bmatrix} \vec{\varphi} \\ 0 \end{bmatrix}$$

$(\underline{c})_J = \int_\Omega b_h^J\,dx \to$ cf. right-hand-side vector

$[\ v \mapsto \int_\Omega f \cdot v\,dx$ with $f \equiv 1\ ]$

get it through "assembleVectorLocally()"

with element vector $\frac{1}{3}|K| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\int_K 1 \cdot \lambda_e\right)_{e=1}^3$

barycentric coordinate function.

How to build $(N+1)\times(N+1)$-Galerkin matrix?

$(N := \dim S_1^0(\mathcal{M}))$

Option :
- Assemble A in COO-Format
- Add triplets for $\underline{c}, \underline{c}^T$
- Convert to CRS-Format

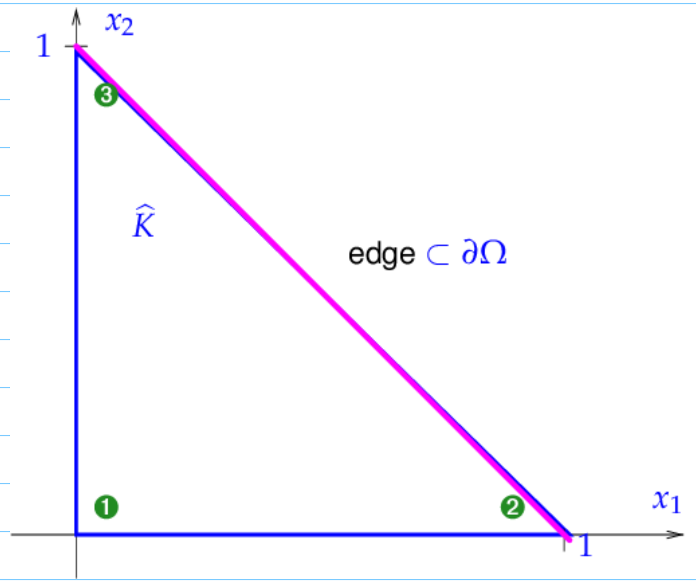
Note : In solution N. dofs $\hat{=}$ $N+1$

③

HW 6.2 d)

$$a(u,v) = \int_\Omega \mathrm{grad}\,u \cdot \mathrm{grad}\,v \, dx + \int_{\partial\Omega} c\,u\,v \, dS$$

$$c > 0$$



Local shape function: $\lambda_1, \lambda_2, \lambda_3$

Compute the $\mathcal{S}_1^0(\mathcal{M})$ element (stiffness) matrix $\mathbf{A}_{\widehat{K}}$ corresponding to $\mathrm{a}(\cdot,\cdot)$ for the unit triangle $\widehat{K}$ with vertices $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. Assume that the edge connecting $\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$ forms part of $\partial\Omega$ and that the coefficient c is constant along this edge.

$$\lambda_1(\underline{x}) = 1 - x_1 - x_2, \quad \lambda_2(x) = x_1, \quad \lambda_3(x) = x_2$$

$$\mathrm{grad}\,\lambda_1(x) = -\begin{bmatrix}1\\1\end{bmatrix}, \quad \mathrm{grad}\,\lambda_2(x) = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathrm{grad}\,\lambda_3 = \begin{bmatrix}0\\1\end{bmatrix}$$

↳ constant vector

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$\int_{\partial\Omega} c\,u\,v\,dS$:

Parametrization of edge e

$$\Pi : \tau \to \begin{bmatrix}\tau \\ 1-\tau\end{bmatrix}, \quad 0 \le \tau \le 1$$

$$\int_e \varphi(x)\,dS(x) = \int_0^1 \varphi(\Pi(\tau)) \underbrace{\left\| \frac{\partial \Pi}{\partial \tau}(\tau) \right\|}_{\sqrt{2}} d\tau$$

Edge contributions:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \int_e \lambda_2^2\,dS & \int_e \lambda_2\lambda_3\,dS \\ 0 & \int_e \lambda_2\lambda_3\,dS & \int_e \lambda_3^2\,dS \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{2} & \frac{1}{6}\sqrt{2} \\ 0 & \frac{1}{6}\sqrt{2} & \frac{1}{3}\sqrt{2} \end{bmatrix}$$

because $\lambda_1|_e = 0$

$$\lambda_2(\Pi(\tau)) = \tau, \quad \lambda_3(\Pi(\tau)) = 1-\tau$$

$$\int_e \lambda_2\lambda_3\,dS = \int_0^1 \tau(1-\tau)\sqrt{2}\,d\tau = \frac{1}{6}\sqrt{2}$$

Q: RK-SSM for MOL-ODE: Stability

$$M\dot{\vec{\mu}} + A\vec{\mu} = \vec{\varphi}(t) \qquad (1)$$

- RK-SSM for $\dot{y} = \alpha y$ spawns discrete evolution operator $\Psi^{\tau} y = S(\tau\alpha)\, y$

  $\uparrow$ stability function

- Diagonalization [$\leftrightarrow$ change of basis]

  (1) $\rightarrow$ decoupled scalar ODEs

  $$\dot{\eta}_j(t) + \lambda_j \eta_j(t) = \tilde{\varphi}_j(t) \qquad (2)$$

  $\uparrow$ generalized EV: $A\vec{\xi} = \lambda M\vec{\xi}$

  A, M s.p.d. $\Rightarrow \lambda_j > 0$

- RK-SSM commutes with diagonalization

$\triangleright$ No blow-up, if RK-SSM does not blow up for (2)

For $\vec{\varphi} = 0$: no blow-up, if $|S(-\tau\lambda_j)| \leq 1 \quad \forall j$

(If $\vec{\varphi} \neq 0$: growth possible, if $|S(-\tau\lambda_j)| = 1$)

Q: FE GSF
- associated with a mesh entity
- cover a mesh entity

Third main ingredient of FEM: locally supported basis functions
(see Section 2.2 for role of bases in Galerkin discretization)

Basis functions $b_h^1, \dots, b_h^N$ for a finite element trial/test space $V_{0,h}$ built on a mesh $\mathcal{M}$ *must* satisfy:

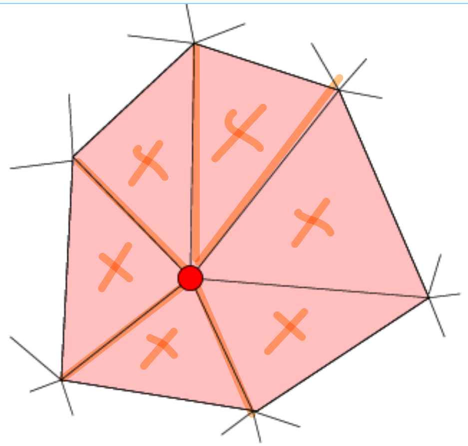
($\mathfrak{B}_1$) $\mathfrak{B}_h := \{b_h^1, \dots, b_h^N\}$ is basis of $V_{0,h}$ ➤ $N = \dim V_{0,h}$,

($\mathfrak{B}_2$) each b_h^i is associated with a single geometric entity (cell/edge/face/vertex) of $\mathcal{M}$,

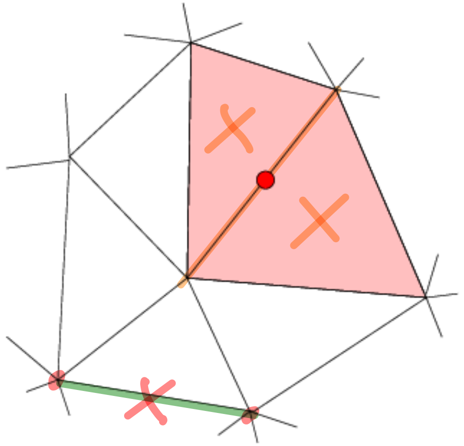
($\mathfrak{B}_3$) $\operatorname{supp}(b_h^i) \subset \bigcup\{\overline{K}: K \in \mathcal{M}, \boldsymbol{p} \subset \overline{K}\}$, if b_h^i associated with cell/edge/face/vertex $\boldsymbol{p}$.

$\downarrow$ the entity on which the interpolation node of b_h^j is located (for cardinal basis)

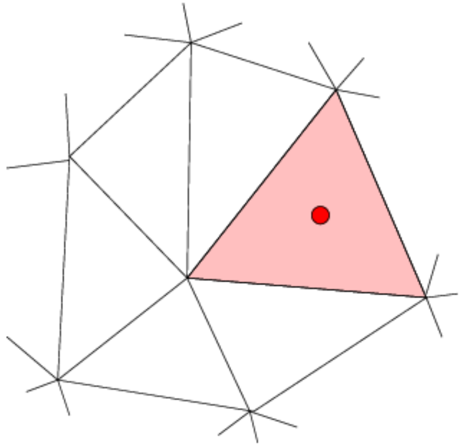
b_h^j covers the mesh entity E, if $b_h^j|_E \neq 0$



Support of node-associated basis function, *cf.* Fig. 96



Support of edge-associated basis function



Support of cell-associated basis function

HW 3.8 d)

$$a(u,v) := \int_\Omega \mathbf{grad}\,u\cdot\mathbf{grad}\,v\,dx + \int_{\Gamma_0} uv\,dS$$

$\mathcal{S}_1^0(\mathcal{M}_\ell)$-FE solution

By optimality of Galerkin solution

$$\|u-u_h\|_a \le \|u-v_h\|_a \quad \forall v_h \in V_{0,h}$$

$$\le \|u-I_\ell u\|_a$$

$\in H^2(\Omega)$

p.w. linear interpolation

$$\|u-I_\ell u\|_a^2 = \underbrace{\|\mathbf{grad}(u-I_\ell u)\|^2_{L^2(\Omega)}}_{=O(h_\mathcal{M}^2)} + \underbrace{\|u-I_\ell u\|^2_{L^2(\partial\Omega)}}_{=O(h_\mathcal{M}^3)}$$

Theorem 3.3.2.21. Error estimate for piecewise linear interpolation

For any $u \in C^2(\bar{\Omega})$ and 2D piecewise linear interpolation $\mathsf{I}_1 : C^0(\overline{\Omega}) \to \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh, holds

$$\|u-\mathsf{I}_1 u\|_{L^2(\Omega)} \le \sqrt{\tfrac{3}{8}}\, h_\mathcal{M}^2 \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)},$$

$$\|\mathbf{grad}(u-\mathsf{I}_1 u)\|_{L^2(\Omega)} \le \sqrt{\tfrac{3}{24}}\, \rho_\mathcal{M} h_\mathcal{M} \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)}.$$

where $h_\mathcal{M}$ denotes the mesh width ($\to$ Def. 3.2.1.4) and $\rho_\mathcal{M}$ the shape regularity measure ($\to$ Def. 3.3.2.20) of $\mathcal{M}$.

6 regularity of $u|_{\partial\Omega}$

$$\|u - I_\ell u\|^2_{L^2(\partial\Omega)} \le C \underbrace{\|u - I_\ell u\|_{L^2(\Omega)}}_{O(h_m^2)} \underbrace{\|u - I_\ell u\|_{H^1(\Omega)}}_{O(h_m)}$$

Theorem 1.9.0.10. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \le C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,.$$