ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

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Q & A Session

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[HW 3.5, broken recording]

Discrete maximum principle :

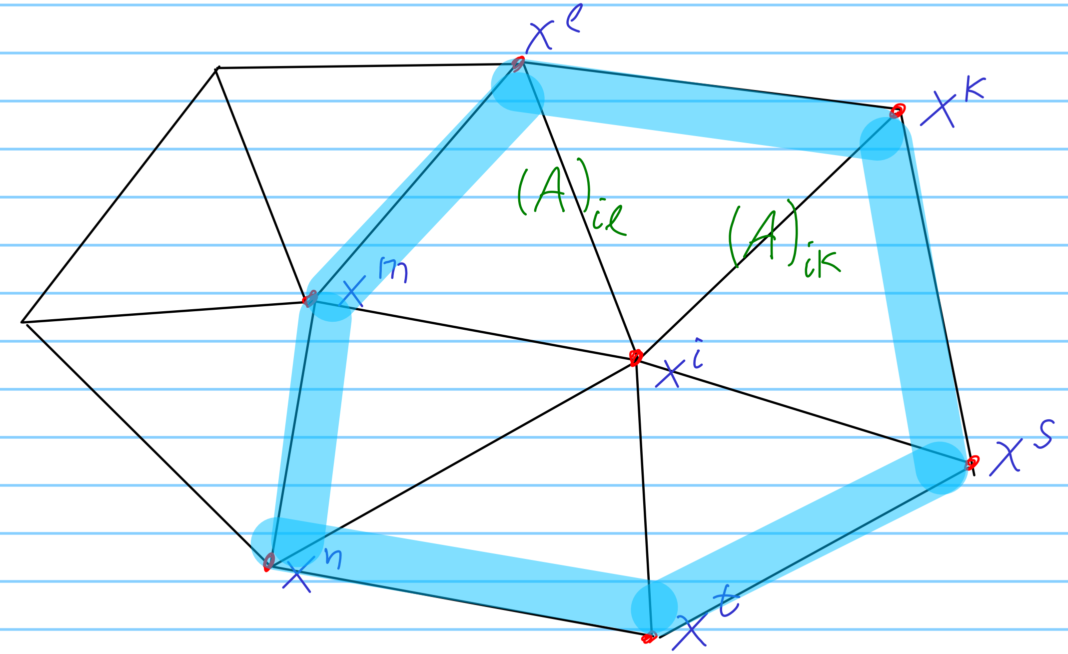
Theorem 3.7.1.2. Maximum principle for 2nd-order elliptic BVP

For $u \in C^0(\overline{\Omega}) \cap H^1(\Omega)$ holds the maximum principle

$$-\operatorname{div}(\kappa(\boldsymbol{x})\,\mathbf{grad}\,u) \geq 0 \quad \Longrightarrow \quad \min_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x}) = \min_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x}),$$

$$-\operatorname{div}(\kappa(\boldsymbol{x})\,\mathbf{grad}\,u) \leq 0 \quad \Longrightarrow \quad \max_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x}).$$

Averaging argument & stencil notation



$S_1^0(\mathcal{M})$ - FE solution $\Longleftrightarrow$: $\vec{\mu} \in \mathbb{R}^N$

$(\vec{\mu})_i = \mu_i = u_h(\boldsymbol{x}^i)$, $\boldsymbol{x}^i \in \mathcal{V}(\mathcal{M})$

▷ i-th equation of Galerkin LSE

$$\underbrace{(A)_{ii}}_{>0}\mu_i + (A)_{il}\mu_l + (A)_{ik}\mu_k + \dots = 0$$

[assuming $f \equiv 0$]

$$\mu_i = -\frac{1}{(A)_{ii}}\left((A)_{il}\mu_l + (A)_{ik}\mu_k + \dots\right)$$

Recall: $a(u,v) = \int_\Omega \kappa(x)\,\mathrm{grad}\,u \cdot \mathrm{grad}\,v \,dx$

If x^i is an interior vertex: use test function

$v_h \in S_1^0(\mathcal{M})$: $v_h \equiv 1$ on

[$v_h := b_h^i + b_h^k + b_h^l + b_h^m + \dots + b_h^s$]

▷ $\mathrm{grad}\, v_h = 0$ on

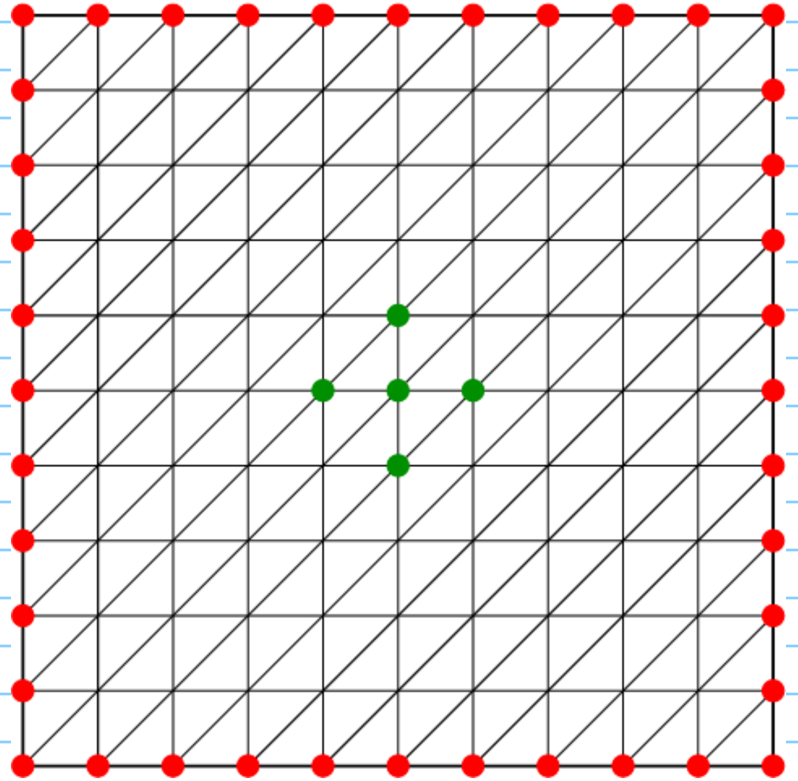
▷ $a(b_h^i, v_h) = 0$

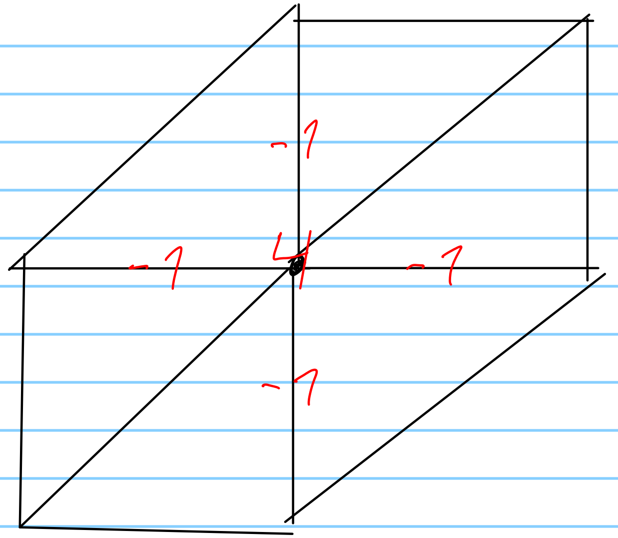
$$a(b_h^i, b_h^i) + a(b_h^i, b_k^k) + \dots + a(b_h^i, b_h^s) = 0$$

▷ $(A)_{ii} + (A)_{ik} + \dots + (A)_{is} = 0$

$$\mu_i = \frac{1}{\sum -(A)_{ix}}\left(-(A)_{il}\mu_l - (A)_{ik}\mu_k - \dots\right)$$

Special situation





Concept of (arithmetic) mean:

$y = \frac{1}{\sum \omega_e}\sum \omega_e x_e$ is an average

↳ weights

if $\omega_e \geq 0$, $\sum \omega_e > 0$

▷ $\min\{x_e\} \leq y \leq \max\{x_e\}$

$$\mu_i = \frac{1}{\sum -(A)_{i*}} \left(-(A)_{il}\mu_l - (A)_{ik}\mu_k - \dots \right)$$

If $(A)_{i,*} \leq 0$ then μ_i is an arithmetic mean of the values in the surrounding nodes.

$$\min\{\mu_*\} \leq \mu_i \leq \max\{\mu_*\}$$

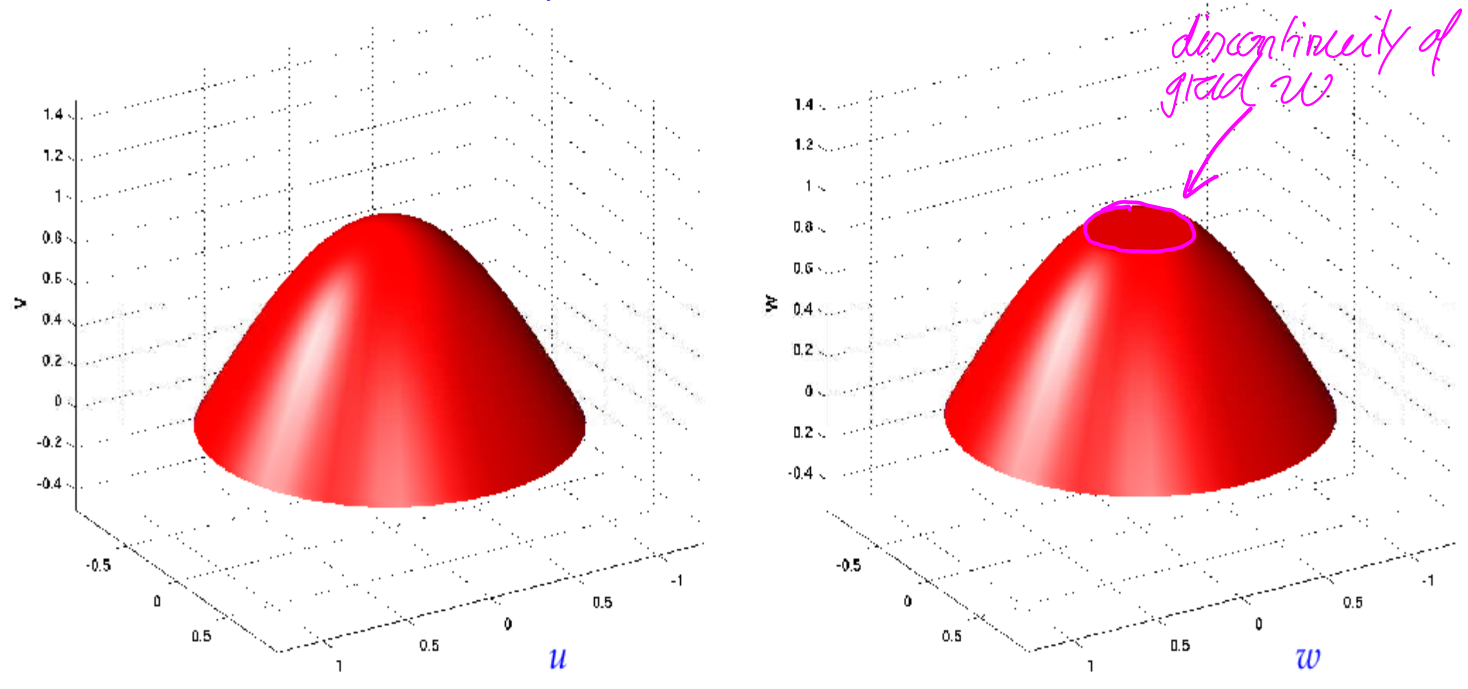
▷ Rules out strict extremum of u_h in x^i.

- $(\widetilde{\mathbf{A}})_{ii} > 0$ (positive diagonal) , (3.7.2.7)
- $(\widetilde{\mathbf{A}})_{ij} \leq 0$ for $j \neq i$ (non-positive off-diagonal entries) , (3.7.2.8)
- $\sum_j (\widetilde{\mathbf{A}})_{ij} = 0$, if x^i is interior node . (3.7.2.9)

$$\ulcorner \int_0^1 \log(x)\,dx = [x\log x - x]_0^1 = 1 < \infty \lrcorner$$

④

$$u = \underset{\substack{v \in H^1(\Omega) \\ v = g \text{ on } \partial\Omega}}{\operatorname{argmin}} \int_\Omega \kappa(x) \|\operatorname{grad} v\|^2 \, dx$$



Since the "flat top" of w does not contribute to the energy functional J, whereas u makes a positive contribution in that zone, we conclude

$$\int_\Omega \kappa(\boldsymbol{x}) \|\mathbf{grad}\, u(\boldsymbol{x})\|^2 \, d\boldsymbol{x} \geq \int_\Omega \kappa(\boldsymbol{x}) \|\mathbf{grad}\, w(\boldsymbol{x})\|^2 \, d\boldsymbol{x}\,.$$

LehrFEM++ functions for enforcing Dirichlet b.c.

$$\begin{aligned} u_h &\in S_1^0(\mathcal{M}) \\ u_h &= g_h \text{ on } \partial\Omega \end{aligned} : \quad \int_\Omega \kappa(x) \operatorname{grad} u_h \cdot \operatorname{grad} v_h \, dx = \int f v_h \, dx \quad \forall v_h \in S^0_{1,0}(\mathcal{M})$$

$$A = \left[\begin{array}{c|c} A_{ii} & A_{i\partial} \\ \hline A_{i\partial} & A_{\partial\partial} \end{array}\right] \hat{=} \; S_1^0(\mathcal{M}) \text{ - Galerkin matrix}$$

interior shape function ; shape functions on boundary

$$\begin{bmatrix} A_{ii} & A_{i\partial} \\ A_{i\partial} & A_{\partial\partial} \end{bmatrix} \begin{bmatrix} \vec{\mu}_i \\ \vec{\mu}_\partial \end{bmatrix} = \begin{bmatrix} \vec{\varphi}_i \\ \vec{\varphi}_\partial \end{bmatrix}$$

$\vec{\mu}_\partial$ ↳ given by g_h

$$A_{ii} \vec{\mu}_i = \vec{\varphi}_i - A_{i\partial} \vec{\mu}_\partial$$

fix_flagged_solution_comp:

$$\begin{bmatrix} A_{ii} & A_{i\partial} \\ A_{i\partial} & A_{\partial\partial} \end{bmatrix} \longrightarrow \begin{bmatrix} A_{ii} & 0 \\ 0 & I \end{bmatrix}$$

Fix_..._alt():

$$\begin{bmatrix} A_{ii} & A_{i\partial} \\ 0 & I \end{bmatrix} \begin{bmatrix} \vec{\mu}_i \\ \vec{\mu}_* \end{bmatrix} = \begin{bmatrix} \vec{\varphi}_i \\ \vec{\mu}_\partial \end{bmatrix} \rightarrow \vec{\mu}_* = \vec{\mu}_\partial$$

$$A_{ii}\vec{\mu}_i = \vec{\varphi}_i - A_{i\partial}\vec{\mu}_\partial$$