ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

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Q & A Session

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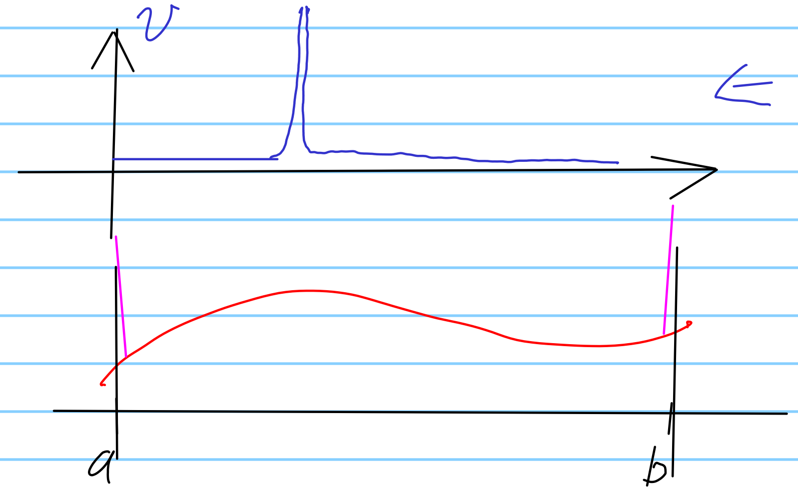
1) Guidance to h.w. problems in tutorial videos
2) Links to video files in PDF
3) Sub-folders in polybox

Norms:

$\|\cdot\| \hat{=}$ Euclidean norm

$\|v\|_0 = \|v\|_{L^2(\Omega)} = \left(\int_\Omega |v|^2 \, dx\right)^{1/2} \hat{=}$ average size of v

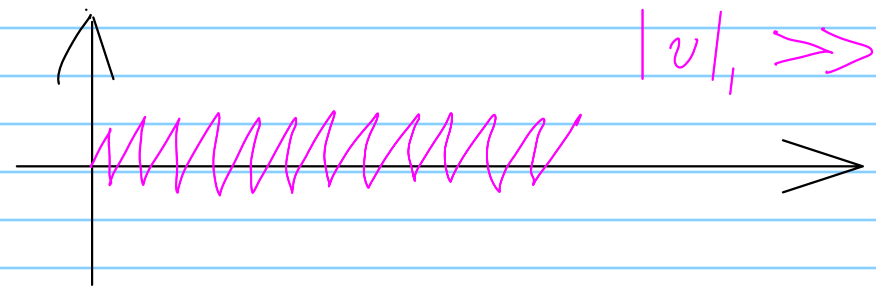
↳ "blind to local effects"



$\Leftarrow \|v\|_0$ small

$$|v|_1 = |v|_{H^1(\Omega)} = \left(\int_\Omega \|\operatorname{grad} v\|^2 \, dx \right)^{1/2}$$

$\triangleq$ measures strength of local variations



$|v|_1 \gg 1$, $\|v\|_0 \ll 1$

$\|\cdot\|_\infty \triangleq$ maximum norm

"blind to global behavior"

- $|\cdot|_1$ vs $\|\cdot\|_1 := (|\cdot|_1^2 + \|\cdot\|_0^2)^{1/2}$

  $\hookrightarrow$ semi-norm : $= 0$ for constants $\Rightarrow$ no norm

Seek : $u \in H^1(\Omega)$, but $u \notin C^1_{pw}(\overline{\Omega})$

$\Omega := \{x : \|x\| < e\}$: $u(x) = \log|\log\|x\||$

$u(0) = \infty$ and $\|u\|_1 < \infty$

Connection :

$$u \in H^1_0(\Omega): \underbrace{\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \, dx}_{\downarrow} = v(0) \quad \forall v \in H^1_0(\Omega)$$

test function

$=: \ell(v)$

Energy space $H^1(\Omega)$

LVP: $u \in V_0 : a(u,v) = \ell(v) \quad \forall v \in V_0$

$$\Updownarrow$$

QMP: $\frac{1}{2} a(u,u) - \ell(u) \longrightarrow \min$

Necessary for existence of a minimizer :

$\exists C > 0 : \quad |\ell(v)| \leq C \, a(v,v)^{1/2} = C \|v\|_a$

Energy norm^2 $\Leftrightarrow$ energy

LVP : $u \in V_0 \quad \underbrace{a(u,v)}_{\text{s.p.d.}} = \ell(v) \quad \forall v \in V_0$

$$\boxed{\|v\|_a := a(v,v)^{1/2}}$$

Membrane model : $\frac{1}{2}a(u,u) \hat{=}$ elastic energy of membrane "with shape u"

Ad p. 3/1.4

$$J(v) = \frac{1}{2}\underbrace{a(v,v)}_{\text{s.p.d.}} - \underbrace{\ell(v)}_{\text{cont. w.r.t. } \|\cdot\|_a} + c \;, v \in V_0$$

$\Leftrightarrow$ LVP : $u^* \in V_0$: $a(u^*,v) = \ell(v) \quad \forall v \in V_0$

$$J(v) = \frac{1}{2}a(v,v) - a(u^*,v) + \frac{1}{2}a(u^*,u^*) - \frac{1}{2}a(u^*,u^*) + c$$
$$= \frac{1}{2}a(u^*-v, u^*-v) - \frac{1}{2}a(u^*,u^*) + c$$
$$= \frac{1}{2}\|u^*-v\|_a^2 \underbrace{- \frac{1}{2}a(u^*,u^*) + c}_{\text{minimal value}}$$

Ch 1 : • on mathematical models centering around energy minimization on function spaces [Sect 1.2]

• QMP $\Longleftrightarrow$ LVP

dim $< \infty$: $J(\vec{v}) = \frac{1}{2}\vec{v}^T A\vec{v} - \vec{b}\cdot\vec{v}$

$\operatorname{grad} J(\vec{u}) = A\vec{u} - \vec{b} \overset{!}{=} 0 \hat{=}$ LSE