ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

~~Numerical Methods for Partial Differential Equations~~

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Welcome & Introduction

Feb 22, 2021

I Topic

Numerical Methods for continuum models with local interaction

continuum : unknowns are functions* $u = u(x)$, $u = u(x,t)$: ∞-dimensional spaces

* also fields

models : from fluid mechanics, electromagnetics, quantum chemistry, quant. finance

$\rightarrow$ has context, has structure

PDE $\neq$ just equation $F(u, Du) = 0$

Du ← some derivatives

$\downarrow$ should be preserved by numerical method

Example: Heat equation :

$$u = u(x,t) : \quad \frac{\partial u}{\partial t} - \Delta u = f(x,t)$$

$\rightarrow$ Dissipative evolution towards equilibrium

(Stationary, incompressible) Navier-Stokes equations:

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + D\mathbf{u}\cdot\mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} \quad \text{in } \Omega\,,\\
\operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega\,, \leftarrow \text{incompressibility}\\
\mathbf{u} &= 0 \quad \text{on } \partial\Omega\,.
\end{aligned}
\tag{0.4.3.1}
$$

ν $\hat{=}$ dynamic viscosity
$\mathbf{f}:\Omega \mapsto \mathbb{R}^3$ $\hat{=}$ given external force field
$\mathbf{u}:\Omega \mapsto \mathbb{R}^3$ $\hat{=}$ velocity field (unknown)
$p:\Omega \mapsto \mathbb{R}$ $\hat{=}$ pressure (unknown)

II. Prerequisites ▷ Sect. 0.3.

III. Contents and Outline

I. Equilibrium models (Elliptic boundary-value problems)

(Ex. : $\Delta u = f$ in $\Omega \subset \mathbb{R}^d$ + $\underbrace{u = 0 \text{ on } \partial\Omega}_{\text{boundary cond.}}$)

- Theory : variational approach
- Finite-element method :
  - Foundations
  - Algorithms
  - Implementation

II. Evolution problem in finite-dimensional state space

(Initial-value problems for ODEs)

$$\dot{y} = f(y) \quad , \quad y(0) = y_0 \in \mathbb{R}^n$$

- single-step method
- stability theory

III. Linear evolution problems for fields

- Parabolic initial-boundary-value problems

Ex : Heat equation

IV. Non-linear conservation laws (in 1D)

$u = u(x,t)$: $\frac{\partial u}{\partial t} + \partial_x f(u) = 0$

Focus :
- not on theory
- not on recipes
- on algorithms (→ Coding projects)
- on behavior / scope / limitations of methods

Workload: $300\,h = \underbrace{200\,h}_{\text{during term}} + \underbrace{100\,h}_{\text{exam prep.}}$

(12-15 h / week)

IV. Flipped-Classroom Teaching Format

A flipped-classroom course

This course will follow the **flipped-classroom** paradigm:

Learning by **self-study** guided by

[weekly] instruction videos tablet notes	lecture notes	interactive Q&A sessions	homeworks tutorial classes

Defines course contents → reference

ask questions in Moodle Fora

Use pause button!

(3-5 h / week)

(5-7 h / week)

Report errors / make suggestions through Moodle Fora!

Flipped-classroom advice

- Develop a routine
- Stable setting
- Take breaks
- Group work

①

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session March 1

Q1 : Review question 1.4.3.8.

A) State the linear variational problem equivalent to the minimization of a quadratic functional → $J : \mathbb{R}^2 \to \mathbb{R}$, $J(\boldsymbol{x}) := \|\boldsymbol{x}\|_2^2 + x_1 + x_2 + 1$.

Definition 1.2.3.2. Quadratic functional

A quadratic functional on a *real vector space* V_0 is a mapping $J : V_0 \mapsto \mathbb{R}$ of the form

$$J(u) := \tfrac{1}{2}\mathsf{a}(u,u) - \ell(u) + c\,, \quad u \in V_0\,, \tag{1.2.3.3}$$

where $\mathsf{a} : V_0 \times V_0 \mapsto \mathbb{R}$ is a symmetric bilinear form (→ Def. 0.3.1.4), $\ell : V_0 \mapsto \mathbb{R}$ a linear form, and $c \in \mathbb{R}$.

$$J(\underline{x}) = \underbrace{\underline{x}^T\underline{x}}\ + \underbrace{\begin{bmatrix}1\\1\end{bmatrix}^T\underline{x}} + 1 \quad , \quad \underline{x} \in \mathbb{R}^2$$

$$\leftrightarrow\ a(\underline{x},\underline{y}) = 2\,\underline{x}^T\underline{y} \qquad \ell(\underline{x}) = -\begin{bmatrix}1\\1\end{bmatrix}\cdot\underline{x}$$

Theorem 1.4.1.8. Equivalence of quadratic minimization problem and linear variational problem

For a (generalized) quadratic functional $J(v) = \frac{1}{2}\mathsf{a}(v,v) - \ell(v) + c$ on a vector space V and with a symmetric positive definite bilinear form $\mathsf{a} : V \times V \to \mathbb{R}$ are equivalent:

(i) *The quadratic minimization problem for J has unique minimizer $u_* \in \widehat{V}$ over the affine subspace $\widehat{V} = g + V_0$, $g \in V$.* [$\widehat{V}$ ≙ affine space]

(ii) *The linear variational problem*

$$u \in \widehat{V}: \quad \mathsf{a}(u,v) = \ell(v) \quad \forall v \in V_0\,,$$

has a unique solution $u_* \in \widehat{V}$.

LVP here: $x \in \mathbb{R}^2: \quad 2x^T y = -[1,1]^T y \quad \forall y \in \mathbb{R}^2$

Note: $V_0 = V = \mathbb{R}^2, \quad g = 0$

B)

Let V be a vector space, $\mathsf{a} : V \times V \to \mathbb{R}$ a symmetric positive definite bilinear form and $\ell : V \to \mathbb{R}$ a linear form bounded w.r.t. to the energy norm $\|\cdot\|_{\mathsf{a}}$ induced by $\mathsf{a}(\cdot;\cdot)$. Assuming that the quadratic functional $(\|v\|_a^2 := a(v,v))$

$$J(v) := \tfrac{1}{2}\mathsf{a}(v,v) - \ell(v) + c\,, \quad v \in V\,, \tag{1.2.3.3}$$

Thm. 1.4.1.8

has a minimizer, what is the $\underbrace{\text{minimal value of } J}_{=J(u)}$ expressed

- in terms of $\|u\|_{\mathsf{a}}$, or
- in terms of $\ell(u)$,

where $u \in V$ satisfies ($\rightarrow u$ is the minimizer of J)

LVP: $\mathsf{a}(u,v) = \ell(v) \quad \forall v \in V\,.$

$$J(u) = \tfrac{1}{2} a(u,u) - \ell(u) + c \overset{(LVP)}{=} -\tfrac{1}{2}\|u\|_a^2 + c = -\tfrac{1}{2}\ell(u) + c$$

C) Sobolev space of functions vanishing on $\partial\Omega$

For a bounded domain $\Omega \subset \mathbb{R}^2$ consider the second-order linear elliptic Dirichlet problem

LVP: $u \in H_0^1(\Omega)$: $\underbrace{\int_\Omega (\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u(\boldsymbol{x})) \cdot \mathbf{grad}\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{=:a(u,v)} = \underbrace{\int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{=:\ell(v)} \quad \forall v \in H_0^1(\Omega)\,.$

with uniformly positive definite $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^{2,2}$.

1. For what source functions f will the right-hand side functional still be continuous on $H_0^1(\Omega)$? (Give a reasonably general sufficient condition.)
2. Let $u \in H_0^1(\Omega)$ be the solution of the above variational problem. For which norms introduced in Section 1.3 ($\square \in \{H^1(\Omega), L^2(\Omega)\}$) does the estimate

$$\|u\|_\square \leq C\|f\|_\square$$

hold with a constant $C > 0$ independent of f?

1) To show: Continuity / boundedness of ℓ for "suitable" f $= f \in L^2(\Omega)$

(*) $\exists C > 0: \; |\ell(v)| \leq C \|v\|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega)$

Tool: Cauchy-Schwarz inequality (CSI)

$$|\ell(v)| = \left|\int_\Omega f(x)v(x)\,dx\right| \overset{(CSI)}{\leq} \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}$$

since $\|v\|_{H^1(\Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \|\mathrm{grad}\, v\|_{L^2(\Omega)}^2$

Why is $(*)$ important?

$\rightarrow$ Necessary for existence of solutions/minimizer of associated QMP

[Lemma 1.2.3.39]

Recall Ex. 1.2.3.45 (Point load)

Point evaluation $f \in H^1(\Omega) \longrightarrow f(x_0)$, $x_0 \in \Omega$ is not continuous

2) LVP: $V = H_0^1(\Omega)$

$$u \in V: \quad a(u,v) = \ell(v) \quad \forall v \in V$$

- $|\ell(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}$ from 1) (B)
- LVP $\overset{v=u}{\Rightarrow}$ $\|u\|_a^2 = \ell(u)$ (A)
- **Definition 1.2.2.9. Uniformly positive (definite) tensor field**

An matrix-valued function $\mathbf{A} : \Omega \mapsto \mathbb{R}^{n,n}$, $n \in \mathbb{N}$, is called uniformly positive definite, if

$$\exists \alpha^- > 0: \quad (\mathbf{A}(\mathbf{x})\mathbf{z}) \cdot \mathbf{z} \geq \alpha^- \|\mathbf{z}\|^2 \quad \forall\, \mathbf{z} \in \mathbb{R}^n \tag{1.2.2.10}$$

for *almost all* $\mathbf{x} \in \Omega$, that is, only with the exception of a set of volume zero.

$$z^T \underline{\alpha}(x) z \geq \alpha^- \|z\|_2^2 \qquad \text{(UPD)}$$

$\triangleright$
$$\|u\|_a^2 = \int_\Omega \operatorname{grad} u(x)^T \underline{\alpha}(x) \operatorname{grad} u(x)\, dx$$

$u \in H_0^1(\Omega)$
$$\overset{\text{(UPD)}}{\geq} \alpha^- \int_\Omega \|\operatorname{grad} u(x)\|_2^2\, dx$$
$$\geq C \|u\|_{H^1(\Omega)}^2$$

Theorem 1.3.4.17. First Poincaré-Friedrichs inequality

If $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, *is* bounded, *then*

$$\|u\|_0 \leq \operatorname{diam}(\Omega)\, \|\mathbf{grad}\, u\|_0 \quad \forall u \in H_0^1(\Omega)\,.$$

Norm equivalence: $c\|u\|_{H^1(\Omega)} \leq \|u\|_a \leq C\|u\|_{H^1(\Omega)}$ $\forall u \in H_0^1(\Omega)$

$$\|u\|_{L^2(\Omega)}^2 \leq \|u\|_{H^1(\Omega)}^{\cancel{2}} \overset{\text{(NE)}}{\leq} C'\|u\|_a^2 \overset{(A)}{\leq} C'|\ell(u)|$$
$$\overset{(B)}{\leq} C'\|f\|_{L^2(\Omega)} \cancel{\|u\|_{H^1(\Omega)}}$$

$\triangleright$
$$\|u\|_{H^1(\Omega)} \leq C'\|f\|_{L^2(\Omega)}$$

$\delta \in L^2(\Omega), H^1(\Omega)$ possible

4

||v||_H^1 vs |v|_H^1

by Nico Graf - Sunday, 28 February 2021, 6:33 PM

I don't understand the difference between the norm ||v||_H^1 and the "absolute value?" |v|_H^1.

So it would be great if you can explain the notation |v|_H^1 again.

Definition 0.3.1.10. Norm (on a vector space) $\rightarrow$

A **norm** $\|\cdot\|_V$ on an $\mathbb{R}$-vector space V is a mapping $\|\cdot\|_V : V \mapsto \mathbb{R}_0^+$, such that

(definiteness)	$\|v\|_V = 0 \iff v = 0 \quad \forall v \in V$	(N1)
(homogeneity)	$\|\lambda v\|_V = \lvert\lambda\rvert \|v\|_V \quad \forall \lambda \in \mathbb{R},\ \forall v \in V,$	(N2)
(triangle inequality)	$\|w+v\|_V \leq \|w\|_V + \|v\|_V \quad \forall w, v \in V.$	(N3)

Definition 1.3.4.3. Sobolev space $H_0^1(\Omega)$ $\subset H^1(\Omega)$

The space of integrable functions on Ω with square integrable gradient that vanish on the boundary $\partial\Omega$,

$$V_0 := \{v : \Omega \mapsto \mathbb{R} \text{ integrable: } v = 0 \text{ on } \partial\Omega, \int_\Omega |\mathbf{grad}\, v(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} < \infty\}, \tag{1.3.4.2}$$

is the Sobolev space $H_0^1(\Omega)$ with norm

$$\|\mathrm{grad}\, u\|_{L^2(\Omega)} = |v|_{H^1(\Omega)} := \left(\int_\Omega \|\mathbf{grad}\, v\|^2 \,\mathrm{d}\boldsymbol{x}\right)^{1/2} \longrightarrow \text{not a norm on } H^1(\Omega)$$

✎ Notation: $H_0^1(\Omega)$ ← superscript "1", because first derivatives occur in norm
← subscript "0", because zero on $\partial\Omega$

by 1st P.-F. inequality

Definition 1.3.4.8. Sobolev space $H^1(\Omega)$

The Sobolev space

$$H^1(\Omega) := \{v : \Omega \mapsto \mathbb{R} \text{ integrable: } \int_\Omega |\mathbf{grad}\, v(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} < \infty\}$$

is a normed function space with norm

$$\|v\|_{H^1(\Omega)}^2 := \|v\|_0^2 + |v|_{H^1(\Omega)}^2.$$

Q 1.2.3.50 D)

For what values of $\alpha, \beta, \gamma \in \mathbb{R}$ does the quadratic functional

$$J : \mathbb{R}^2 \to \mathbb{R} \quad , \quad J(\boldsymbol{x}) := \underbrace{\boldsymbol{x}^\top \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \boldsymbol{x}}_{\frac{1}{2} a(x,x)} - \gamma x_1 + \alpha\,,$$

possess a unique minimizer.

$$\triangleright \quad a(x,y) = 2x^T \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} y$$

Theorem 1.2.3.44. Existence of unique minimizer in finite dimensions

Let $J(u) := \frac{1}{2}\mathsf{a}(u,u) - \ell(u) + c$ with symmetric positive definite ($\rightarrow$ Def. 1.2.3.27) bilinear form $\mathsf{a} : V_0 \times V_0 \to \mathbb{R}$ ($\rightarrow$ Def. 0.3.1.4), linear form $\ell : V_0 \to \mathbb{R}$, $c \in \mathbb{R}$, be a quadratic functional on the vector space V_0.
If V_0 has finite dimension, then the quadratic minimization problem ($\rightarrow$ Def. 1.2.3.11)

$$u_* = \operatorname*{argmin}_{u \in V_0} J(u)$$

always possesses a unique solution.

5

$a(x,x)$ s.p.d $\Rightarrow \; a(x,x) > 0 \quad \forall x \neq 0$

$\Leftrightarrow \; \begin{bmatrix}\alpha & 0 \\ 0 & \beta\end{bmatrix}$ is s.p.d.

$\Leftrightarrow \; \underbrace{\alpha, \beta > 0}_{\text{necessary / sufficient for uniqueness}}$

Q 1.3.4.29 A)

Which of the following functions belong to the spaces $L^2(]-1,1[)$ and $H^1(\underbrace{]-1,1[}_{=:\Omega})$, respectively?

1. $f_1(x) = |x|$,
2. $f_2(x) = \log|x|$, $\rightarrow$ " $f_2(0) = -\infty$ "
3. $f_3(x) = \operatorname{sgn}(x)$,
4. $f_4(x) = \sqrt{|x| + x}$.

- $f_1 \in L^2(\Omega)$: ✓ because $f_1 \in C^0([-1,1])$ ✓
- $f_1 \in H^1(\Omega)$: ✓ because $f_1 \in C^1_{pw}([-1,1]) \subset H^1(]-1,1[)$
- $f_2 \in L^2(\Omega)$: ✓ because $\|f_2\|_{L^2(\Omega)} < \infty$

To check $\int_{-1}^{1} f_2^2(x)\,dx = 2\int_0^1 \log^2 x \, dx \overset{z = \log x}{=} 2\int_{-\infty}^{0} z^2 e^z \, dz < \infty$!

- $f_2 \in H^1(\Omega)$: NO

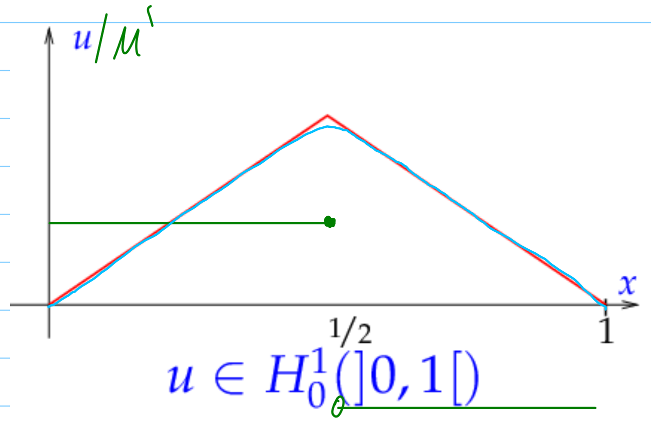
$\int_{-1}^{1} \frac{1}{|x|}\,dx$ " $\longrightarrow \infty$ "

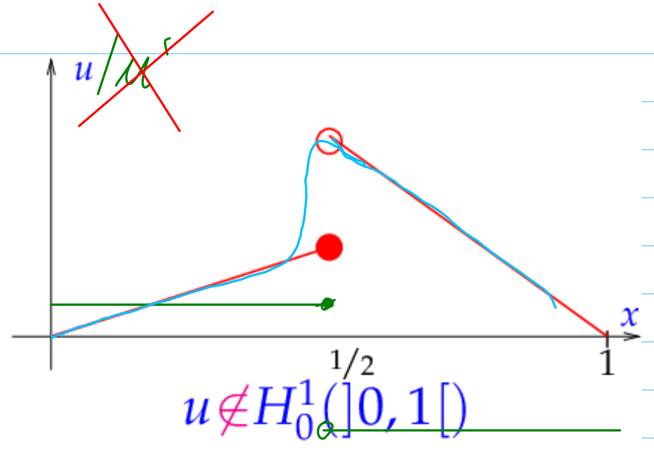
- $f_3 \in L^2(\Omega)$: ✓, because $\|f_3\|_{L^2} = \sqrt{2}$
- $f_3 \in H^1(\Omega)$: NO

does not matter, because we "integrate over it"!

Theorem 1.3.4.23 and example 1.3.4.24
by Basil Ruch - Friday, 26 February 2021, 3:17 PM

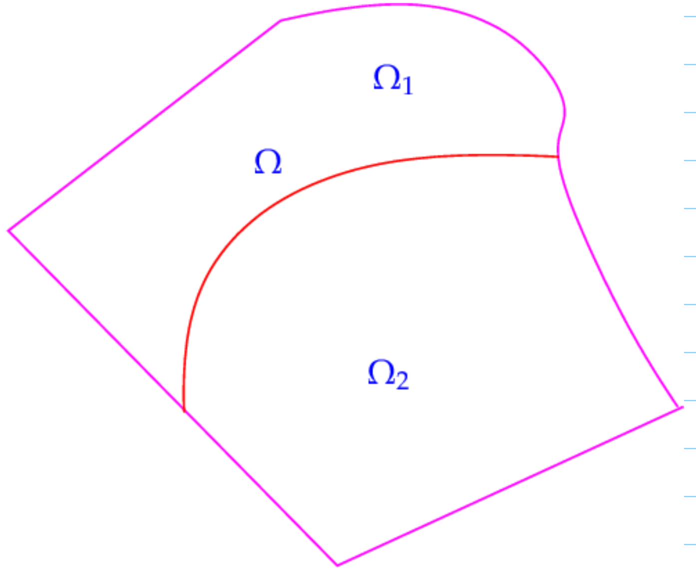
I don't quite understand why functions in H^1 have to be continuous. What is preventing me from integrating the gradient on each subdomain?





Theorem 1.3.4.23. Compatibility conditions for piecewise smooth functions in $H^1(\Omega)$

Let Ω be partitioned into sub-domains Ω_1 and Ω_2. A function u that is continuously differentiable in the closures of both sub-domains, belongs to $H^1(\Omega)$, if and only if u is continuous on Ω.*

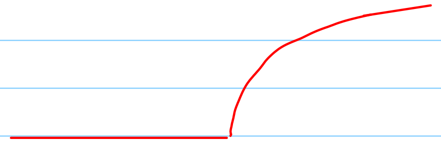


* $u \in C^1(\overline{\Omega}_1)$ and $u \in C^1(\overline{\Omega}_2)$

If u has a discontinuity

$\Rightarrow$ Any sequence $(v_n)_n$ of smooth functions, $v_n(x) \longrightarrow u(x)$ necessarily $\|v_n\|_{H^1} \longrightarrow \infty$

$f_4(x) = \sqrt{x+|x|}$:



- $f_4 \in L^2(\Omega)$: ✓, because $f_4 \in C^0([-1,1])$
- $f_4 \in H^1(\Omega)$: NO

Check, whether p.w. derivative $\in L^2(\Omega)$

$$\int_0^1 ((2\sqrt{x})')^2 dx = \int_0^1 \left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_0^1 \frac{1}{x} dx \text{ "}=\infty\text{"}$$

$$\text{"}\|f_4'\|_{L^2} = \infty\text{"}$$

Note: $H^1(\Omega) \subset L^2(\Omega)$

Technical question:

NPDECODES Git repository:

- → developers
- → homeworks (templates) (← you work here)

How to prevent edited templates from being overwritten when updating non-modified templates?

Faveo Hörold to Everyone 05:56 PM

Isn't this just a case of the user choosing the correct merge directive for the pull request?

```
hint: Pulling without specifying how to reconcile divergent branches is
hint: discouraged. You can squelch this message by running one of the following
hint: commands sometime before your next pull:
hint:
hint: git config pull.rebase false # merge (the default strategy)
hint: git config pull.rebase true # rebase
hint: git config pull.ff only # fast-forward only
hint:
hint: You can replace "git config" with "git config --global" to set a default
hint: preference for all repositories. You can also pass --rebase, --no-rebase,
hint: or --ff-only on the command line to override the configured default per
hint: invocation.
```

or perhaps something along these lines? (Not a git expert, just shooting ideas...)

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

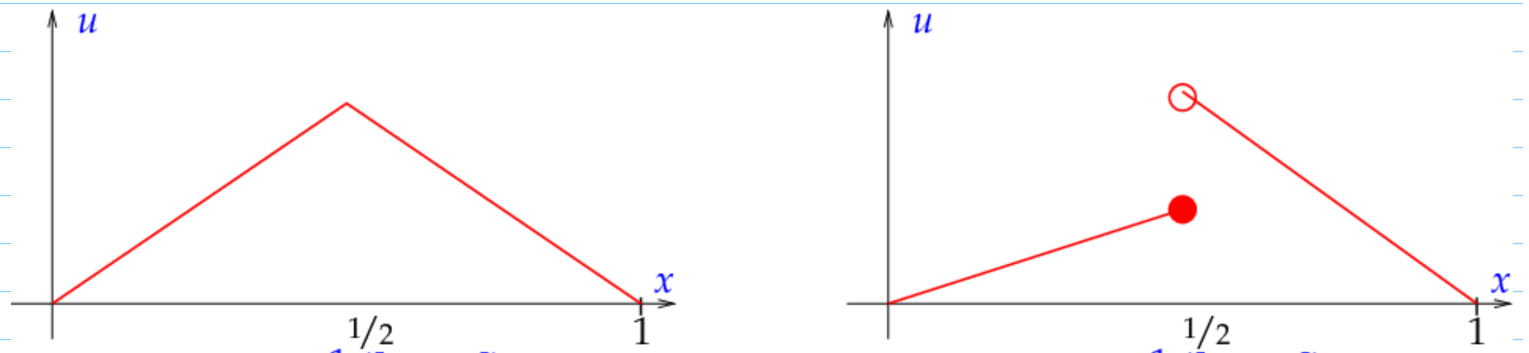
Q & A Session

March 8, 2021

Theorem 1.3.4.23 and example 1.3.4.24

by Basil Ruch - Friday, 26 February 2021, 3:17 PM

I don't quite understand why functions in H^1 have to be continuous. What is preventing me from integrating the gradient on each subdomain?



$u \in H_0^1(]0,1[)$ $\qquad$ $u \notin H_0^1(]0,1[)$

$\longrightarrow$ Q & A March 1

2

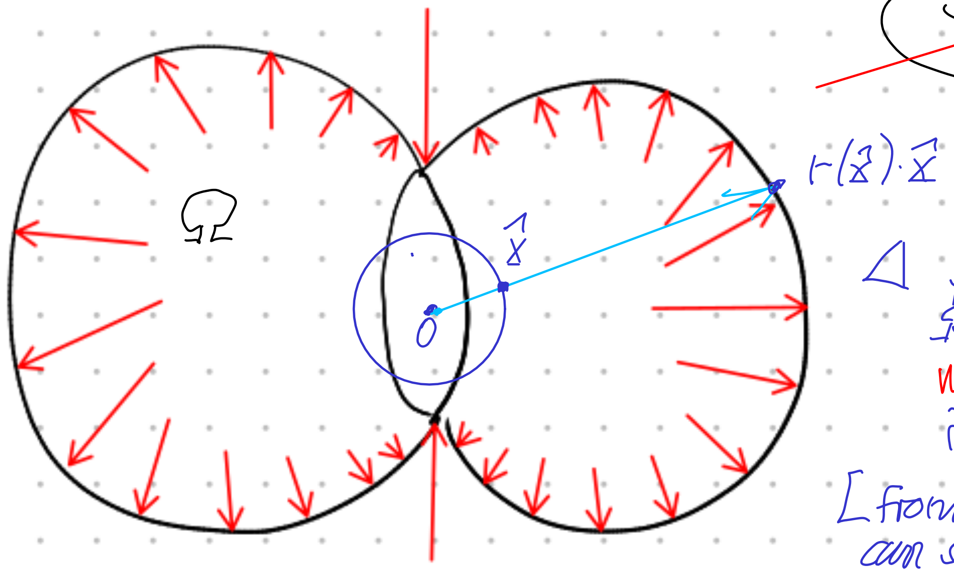
Q.1.2.1.30B :

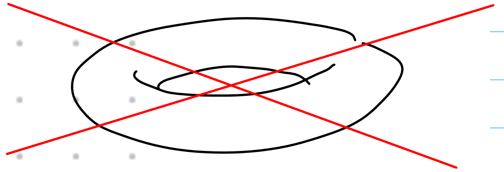
What is a suitable configuration space for an elastic balloon filled with pressurized gas?

Q1.2.1.30.B Balloon shapes not covered by a spherical displacement function
by Faveo Hörold - Wednesday, 24 February 2021, 5:10 PM

Regarding review question 1.2.1.30.B:

It seems to me like a few cases would lead to balloon shapes that cannot be described by the suggested displacement function on a sphere. Consider the following discontinuous forces on a balloon:





Δ possible when Ω is star-shaped w.r.t. a point inside.
[From that point one can see all of $\partial\Omega$]

$\Omega \hat{=}$ volume occupied by balloon

$$\Omega = \left\{ \underline{x} \in \mathbb{R}^3 : \|x\| < r\left(\frac{x}{\|x\|}\right) \right\}$$

for some r : sphere $\longrightarrow \mathbb{R}^+$ continuous

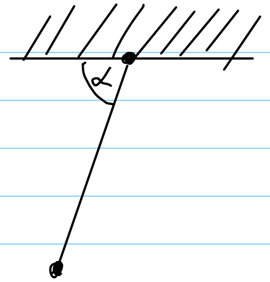
Configuration Spaces
by Gioele Molinari - Monday, 1 March 2021, 9:20 AM

Hello,

I have some questions about configuration spaces.

1. • Is a configuration space always a function space? If not, could you provide an example of such a space? (In the videos are presented only function spaces (string & membrane).)
2. • If we have a function space (as configuration space) must this always be affine?

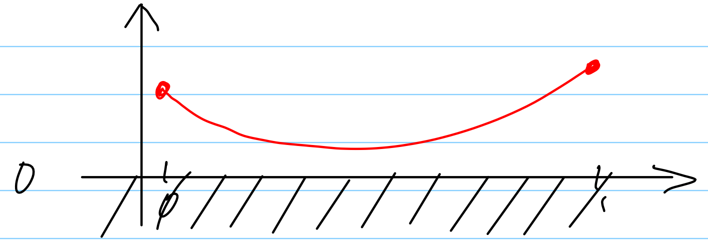
1 No :



$\alpha \in [-\pi/2, \pi/2]$

1D configuration space

2

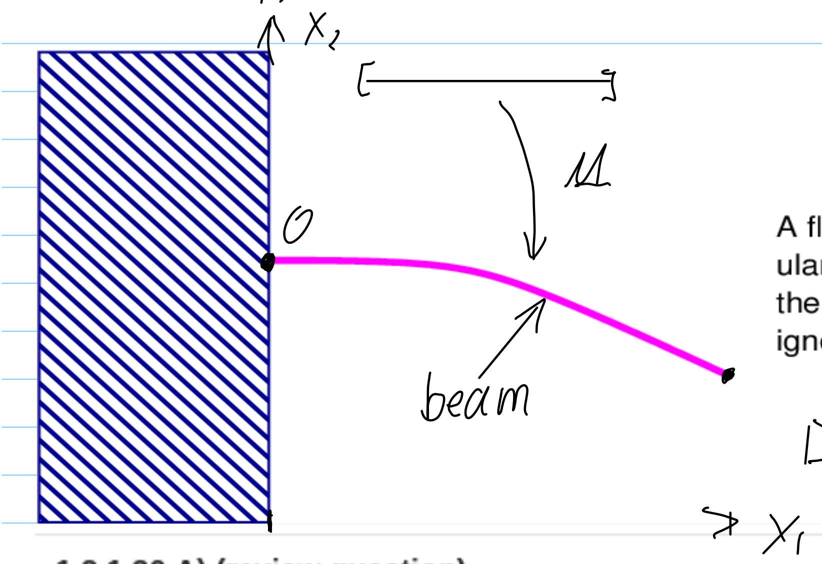


$$V := \{ u : [0,1] \to \mathbb{R},\ u \geq 0 \}$$

↑

No affine space !

Q 1.2.1.30A



A flexible thin beam is mounted on a wall perpendicularly. What is a suitable configuration space for it, if the lateral extension (thickness) of the beam can be ignored?

▷ No graph description

1.2.1.30 A) (review question)

by Michael Vollenweider - Monday, 1 March 2021, 11:43 AM

Regarding review question 1.2.1.30 A):

If we were to use Euclidean coordinates, then depending on how strongly the rod is bent, the interval on the x-axis in which we would need to define our displacement function is not fixed. Is it enough to allow continuously differentiable piecewise functions? Or do we need to use a different coordinate system where we use angles and the length of the rod as our interval?

$$V = \{ \underline{u} : [0,1] \to \mathbb{R}^2 : \underline{u} \in \underbrace{C^1([0,1])}_{\text{no kinks}} \,, \; \underline{u}(0) = 0, \; \underline{u}'(0) = 0 \}$$

Review Q1.5.3.17.D

by Simon Bolt - Saturday, 6 March 2021, 6:25 PM → Sect. 1.5 / 1.5.1

In this Review Question I ended up with the ODE
$(1 + x^2)\, u'' + (2x + x^2)\, u' = 0$
when looking at the particular case of $v(0) = v(1) = 0$ such that boundary terms vanish.

Then, I also expected to recover some Neumann Boundary Conditions by plugging in the ODE, similar to the 2d case of example 1.5.3.11, but I got stuck at
$v(0) + u'(0)\, v(0) - 2\, u'(1)\, v(1) = 0 \quad \forall\, v \in H^1(]0,1[)$
(the leftover boundary terms from integration by parts)

Could you elaborate on how to recover the Neumann BC in this case?

State the 2-point boundary value problem satisfied by the solution of the variational equation

LVP: $$u \in H^1(]0,1[):\quad \int_0^1 (1+x^2)\frac{du}{dx}(x)\left(\frac{dv}{dx}(x) - v(x)\right) = v(0) \quad \forall v \in H^1(]0,1[)\,.$$

r.h.s functional $\ell(v) = v(0)$: continuous on H^1 [HW 1-3]

Goal: LVP ⇒ BVP, must assume $u \in C^2$

Step I: test with $v \in C_0^\infty(]0,1[)$ + ibp:

$$\int_0^1 (1+x^2)\, u' v' - u' v \, dx = v(0) = 0$$

▷ $$\int_0^1 \left\{ -\frac{d}{dx}\left((1+x^2)u'\right) - u' \right\} v \, dx = 0 \quad \forall v \in C_0^\infty$$

Lemma 1.5.1.13. fundamental lemma of the calculus of variations

Let $f \in C^0_{pw}([a,b])$, $-\infty < a < b < \infty$, *satisfy*

$$\int_a^b f(\xi)v(\xi)\,d\xi = 0 \quad \forall v \in C^\infty([a,b]),\ v(a) = v(b) = 0\,.$$

Then $f \equiv 0$.

$\triangleright$ $\{-\frac{d}{dx}((1+x^2)u') - u'\} = 0$ "PDE" (1)

$$\int_a^b F'(x)\,dx = F(b) - F(a)\,, \tag{1.5.1.6}$$

$$F(x) = f(x)\cdot g(x) \quad \Rightarrow \quad F'(x) = f'(x)g(x) + f(x)g'(x)\,: \tag{1.5.1.7}$$

$$\int_a^b f'(x)g(x) + f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a)\,,$$

$$\int_a^b f(\xi)v'(\xi)\,d\xi = -\int_a^b f'(\xi)v(\xi) + \underbrace{(f(b)v(b) - f(a)v(a))}_{\text{boundary terms}} \quad \forall f, v \in C^1_{pw}([a,b])\,. \tag{1.5.1.8}$$

Step II: test with $v \in C^\infty([0,1])$ & use (1)

ibp: $\int_0^1 (1+x^2)u'v' - u'v\,dx =$

$= \cancel{\int_0^1 -\frac{d}{dx}((1+x^2)u' - u')v\,dx} + 2u'(1)v(1) - u'(0)v(0) = v(0)$

$= 0$ by (1)

$\triangleright$ $2u'(1)v(1) - u'(0)v(0) = v(0)$

$\Rightarrow \begin{cases} u'(1) = 0 \\ u'(0) = -1 \end{cases} \mathrel{\hat{=}} \text{b.c.} \quad (2)$

BVP = (1) & (2)

Exercise (1-7.c & 1-7.d) $\longrightarrow$ BVP for vector fields

by Gioele Molinari - Saturday, 6 March 2021, 11:14 AM

In both proofs of subtask 1-7.c and 1-7.d is used Cauchy Schwarz inequality.

In c) is applied component-wise and in d) is applied directly.

Why this happens? In other words: When we have to use the component-wise application and when we could directly apply CSI?

Energy norm for $\underline{v}: \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\|\underline{v}\|_a^2 := \int_\Omega |\operatorname{div} \underline{v}(x)|^2 + \|\underline{v}(x)\|^2\,dx = \|\operatorname{div} \underline{v}\|_{L^2}^2 + \|\underline{v}\|_{L^2}^2$$

c) $|\ell(\underline{v})| = \left|\int_\Omega \underline{f}(x)\cdot\underline{v}(x)\,dx\right|$ $\qquad \|\cdot\| = \|\cdot\|_2$

C.S. in $\mathbb{R}^2$ $\leq \int_\Omega \|\underline{f}(x)\|\cdot\|\underline{v}(x)\|\,dx$

C.S. in $L^2(\Omega)$: $\leq \|\underline{f}\|_{L^2}\cdot\|\underline{v}\|_{L^2} \leq \|\underline{f}\|_{L^2}\|\underline{v}\|_a$

5

NumPDE@ETHZ

d) $\ell(\mathbf{v}) = \int_{\partial\Omega} \mathbf{v}(x)\cdot n(x)\, dS$

Trick: Gauss theorem

$|\ell(\mathbf{v})| = \left| \int_\Omega 1 \cdot \operatorname{div} \mathbf{v}(x)\, dx \right|$

C.S. in L^2: $\leq \|1\|_{L^2} \cdot \|\operatorname{div} \mathbf{v}\|_{L^2} \leq \|1\|_{L^2} \cdot \|\mathbf{v}\|_a$

Exercise (1-5.d)
by Michael Vollenweider - Saturday, 6 March 2021, 10:30 AM

In exercise (1-5.d) it states "Obviously, the right-hand-side functional of (1.5.1) does not meet these smoothness requirements in the whole interval]0,1[, only on parts of it."

How exactly is that obvious?

Context: LVP $\Rightarrow$ BVP, 2pt - BVP

Need: Smoothness requirements

$$\int_0^1 \sigma(x)\, u' v'\, dx = \int_0^1 f(x) v(x)\, dx$$

Assumption 1.5.1.2. Smoothness required for two-point boundary value problems

The following smoothness requirements have to be satisfied

$$u \in C^2([a,b]) \quad , \quad \sigma \in C^1([a,b]) \quad , \quad f \in C^0([a,b]) \; . \tag{1.5.1.3}$$

Now: $\int_0^1 u'v'\, dx = v(1/2) \quad v \in H_0^1(]0,1[) \quad (*)$

HW 1-3: continuous on $H_0^1(]0,1[)$

But, there is no $f \in C^0([0,1]) : v(1/2) = \int_0^1 f(x) v(x)\, dx$

If there was one $\overset{C.S}{\Rightarrow}$ $\underbrace{|v(1/2)| \leq \|f\|_{L^2} \|v\|_{L^2}}$

not possible, because point eval. is not continuous on L^2!

Q 1.3.5.19 E

Give an example for a linear variational problem on $H_0^1(]0,1[)$ with continuous and s.p.d. bilinear form and continuous right-hand side linear form whose solution will not be the solution of a two-point boundary value problem in the sense of classical calculus.

Hint. "in the sense of classical calculus" implies that the solution u should at least be continuously differentiable: $u \in C^1([0,1])$.

Two cases:

(i) See $(*)$

(ii) Discontinuous coefficient σ:

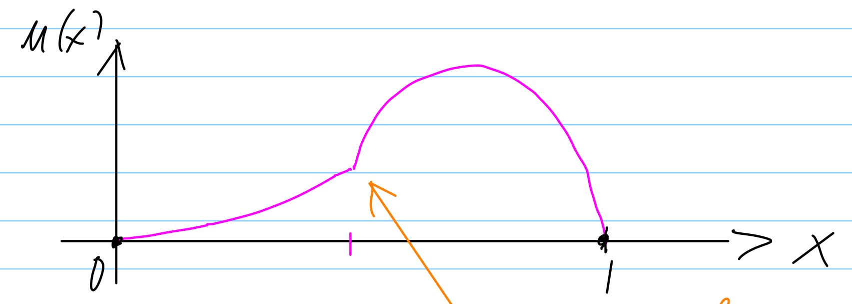
$$\int_0^1 \sigma(x)\, u' v'\, dx = \int_0^1 v(x)\, dx \quad \forall v \in H_0^1(]0,1[)$$

6

$$\sigma(x) = \begin{cases} 1/2\,, & 0 \le x < 1/2 \\ 1\,, & 1/2 \le x \le 1 \end{cases}$$

▷ "DE":
$$\left.\begin{aligned} -\tfrac{1}{2} u'' &= 1 \quad \text{in }]0,1/2[\\ -u'' &= 1 \quad \text{in }]1/2,1[\end{aligned}\right\} \to \text{p.w. parabola}$$

$+$ u continuous



u has a kink at $x = 1/2$

$u \notin C^1([0,1])$

Q. 1.3.5.19H : $\Longrightarrow$ HW 1-7

We consider the variational problem

$$\mathbf{u} : \Omega \to \mathbb{R}^2 : \quad \int_\Omega \operatorname{div} \mathbf{u}(\boldsymbol{x}) \operatorname{div} \mathbf{v}(\boldsymbol{x}) + \mathbf{u}(\boldsymbol{x}) \cdot \mathbf{v}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \quad \forall \mathbf{v} : \Omega \to \mathbb{R}^2 .$$

What is a suitable Sobolev space and what boundary value problem is satisfied by the vector field $\mathbf{u}$?

↳ Energy norm: $\|v\|_a^2 = \|\operatorname{div} v\|_{L^2}^2 + \|v\|_{L^2}^2$

$$V_0 := \{ v : \Omega \to \mathbb{R}^2 : \|v\|_a < \infty \}$$

BVP:

Step I: test with $v \in C_0^\infty(\Omega)$, $v|_{\partial\Omega} = 0$ & i.b.p. (Green's formula)

Theorem 1.5.2.7. Green's first formula → In (*) use with $j \to v$, $v \to \operatorname{div} u$

For all vector fields $\mathbf{j} \in (C^1_{pw}(\overline{\Omega}))^d$ *and functions* $v \in C^1_{pw}(\overline{\Omega})$ *holds*

$$\int_\Omega \mathbf{j} \cdot \mathbf{grad}\, v \, \mathrm{d}\boldsymbol{x} = -\int_\Omega \operatorname{div} \mathbf{j}\, v \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \mathbf{j} \cdot \mathbf{n}\, v \, \mathrm{d}S . \qquad (1.5.2.8)$$

(*) ibp [Remember: $u \in C^2(\overline{\Omega})$]

$$\int_\Omega \operatorname{div} u \cdot \operatorname{div} v + u v \, dx$$
$$= \int_\Omega (-\operatorname{grad} \operatorname{div} u + u)\, v \, dx = 0$$

$\Rightarrow$ PDE: $-\operatorname{grad}\operatorname{div} u + u = 0$ in Ω (1)

Step II: Test with $v \in (C^\infty(\overline{\Omega}))^2$ & use (1)

$$\int_\Omega \operatorname{div} u \cdot \operatorname{div} v + u v \, dx$$
$$= \int_\Omega \underbrace{\cancel{(-\operatorname{grad}\operatorname{div} u + u)}}_{=0} v \, dx - \int_{\partial\Omega} \operatorname{div} u \cdot v \, dS(x) = \int_{\partial\Omega} v \cdot n \, dS$$

⑦

$$\int_{\partial\Omega} (\operatorname{div} \mathbf{u} + 1)\, \mathbf{v}\, dS = 0 \qquad \forall \mathbf{v} \in (C^\infty(\overline{\Omega}))^2$$

Apply a "fundamental lemma on $\partial\Omega$":

$g \in C^0(\partial\Omega)$: $\displaystyle\int_{\partial\Omega} g(x)\, v(x)\, dS(x) = 0 \quad \forall v \in C^\infty(\overline{\Omega})$

$\Rightarrow g = 0$ on $\partial\Omega$

▷ b.c.: $\operatorname{div} \mathbf{u} + 1 = 0$ on $\partial\Omega$

[$\operatorname{div} \mathbf{u}|_{\partial\Omega} \in C^0(\partial\Omega)$ for $\mathbf{u} \in C^1(\overline{\Omega})$, by smoothness assumptions!]

What is the rationale behind the two conclusions in the red boxes (see image)? The requirements for lemma 1.5.3.4 are quite strict and almost never satisfied for the "boundary integral" (being zero on the boundary of the boundary like in example 1.5.3.11)? The first one is an example from the tutorial class and the second is a homework exercise (1-7.f). Thanks for the clarification!

① LVP-to-BVP-example for $u \in H^1(\Omega)$

$$\int_{\partial\Omega} uv\, dS + \int_{\Omega} (1+\|x\|^2)\, \nabla u \cdot \nabla v\, dx = 0 \qquad \forall v \in H^1(\Omega)$$

... $\rightsquigarrow$ plug PDE into the LVP and get

$$\int_{\partial\Omega} [u + (1+\|x\|^2)\, \nabla u \cdot n]\, v\, dS = 0$$

why ? $\rightarrow$ lemma 1.5.3.4 cannot be applied !?

$\Rightarrow u + (1+\|x\|^2)\, \nabla u \cdot n = 0$

② 1-7.f) for $u \in (C^1(\Omega))^2$

$$\int_{\Omega} -\nabla(\alpha \operatorname{div}(u)) \cdot v + u \cdot v\, dx + \int_{\partial\Omega} \alpha \operatorname{div}(u)\, v \cdot n\, dS = \int_{\partial\Omega} v \cdot n\, dS$$

... $\rightsquigarrow$ plug PDE into the LVP and get $\qquad \forall v \in (C^1(\Omega))^2$

$$\int_{\partial\Omega} \alpha \operatorname{div}(u)\, v \cdot n\, dS = \int_{\partial\Omega} v \cdot n\, dS$$

why ?

$\Rightarrow \alpha \operatorname{div}(u) = 1$

Q 1.5.3.19 I:

$1\cdot\left|\frac{\partial u}{\partial x_1}\right|^2 - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} + 1\cdot\left(\frac{\partial u}{\partial x_2}\right)^2$

Which boundary value problem does the minimizer of the functional

$$J(u) = \int_\Omega \left|\frac{\partial u}{\partial x_1}(\mathbf{x}) - \frac{\partial u}{\partial x_2}(\mathbf{x})\right|^2 + |u(\mathbf{x})|^2 - \|\mathbf{x}\| u(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, \quad v \in H_0^1(\Omega)\,,$$

$J(u) = \frac{1}{2}a(u,u) - \underset{\substack{\uparrow\\ \text{linear in } u}}{\ell(u)} + c$

solve? Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain.

$$J(v) = \int_\Omega A\,\mathrm{grad}\, v(x)\cdot\mathrm{grad}\, v(x) + u(x)v(x)\,dx - \ell(v)$$

- $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
- $\ell(v) = \int_\Omega \|x\|\, v(x)\,dx$

$[\ A\underline{x}\cdot\underline{x} = a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2\ ]$

$\triangleright$

$$\text{BVP:}\begin{cases} \text{PDE}: & -\mathrm{div}\, A\,\mathrm{grad}\, u + u = \|x\| & \text{in } \Omega \\ \text{b.c} & u = 0 & \text{on } \partial\Omega \end{cases}$$

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session

March 15, 2021

Follow-up question to the Q&A of 1 march

by Basil Ruch - Monday, 15 March 2021, 8:05 AM

What is the reasoning behind this (red question mark) estimate? (page 3 of the pdf file)

$$\mathbf{z}^T \boldsymbol{\alpha}(\mathbf{x})\mathbf{z} \geq \alpha^- \|\mathbf{z}\|_2^2 \qquad \text{(UPD)}$$

$$\triangleright \quad \|u\|_a^2 = \int_\Omega \operatorname{grad} u(\mathbf{x})^T \boldsymbol{\alpha}(\mathbf{x}) \operatorname{grad} u(\mathbf{x})\,d\mathbf{x}$$

$u \in H_0^1(\Omega)$

$$\overset{\text{(UPD)}}{\geq} \alpha^- \int_\Omega \|\operatorname{grad} u(\mathbf{x})\|_2^2\,d\mathbf{x} \quad \to \|\operatorname{grad}(u)\|_{L^2(\Omega)}^2$$

?

$$\geq C\,\|u\|_{H^1(\Omega)}^2 \qquad \|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\operatorname{grad}(u)\|_{L^2(\Omega)}^2$$

Theorem 1.3.4.17. First Poincaré-Friedrichs inequality

If $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is bounded, then

$$\|u\|_0 \leq \overbrace{\operatorname{diam}(\Omega)}^{=:d} \|\mathbf{grad}\, u\|_0 \quad \forall u \in H_0^1(\Omega).$$

$$\|\operatorname{grad} u\|_{L^2(\Omega)}^2 \geq \tfrac{1}{2}\|\operatorname{grad} u\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\tfrac{1}{d^2}\|u\|_{L^2}^2$$

$$\geq \underbrace{\tfrac{1}{2}\min\{1, 1/d^2\}}_{=:C} \|u\|_{H^1(\Omega)}^2$$

Q 2.3.3.18 G

The use of the composite trapezoidal quadrature rule on a mesh $\mathcal{M}$ of $]a,b[$ with nodes $a = x_0 < x_1 < \dots < x_{M-1} < x_M := b$ amounts to the approximation

$$\text{(TPR)} \qquad \int_a^b \phi(x)\,\mathrm{d}x \approx \tfrac{1}{2}h_1\phi(x_0) + \sum_{j=1}^{M-1}\phi(x_j)\tfrac{1}{2}(h_j+h_{j+1}) + \tfrac{1}{2}h_M\phi(x_M).$$

Explain the difficulty encountered when applying this quadrature rule for the computation of entries of the Galerkin matrix for the linear variational problem

$$u\in H_0^1(]a,b[):\quad \int_a^b \sigma(x)\frac{du}{dx}(x)\frac{dv}{dx}(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x \quad \forall v\in H_0^1(]a,b[),$$

discretized based on $\mathcal{S}_{1,0}^0(\mathcal{M})$ and its tent function basis.

(TPR) requires $\phi(x_j)$ well-defined

However $u_h \in \mathcal{S}_1^0(\mathcal{M}) \Rightarrow \frac{du_h}{dx}$ p.w. constant w.r.t. $\mathcal{M}$

$\Rightarrow \frac{du_h}{dx}$ discontinuous at nodes x_j !

$\Rightarrow \frac{du_h}{dx}(x_j)$ not well-defined

Remedy: use local trapezoidal rule!
(one-sided limits depending on cell)

Q 2.3.3.18 H

We consider the bilinear form

$$\mathrm{b}(u,v) := \int_0^1 u(x)\frac{dv}{dx}(x)\,\mathrm{d}x\,,\quad u\in L^2(]0,1[)\,,\quad v\in H^1(]0,1[)\,.$$

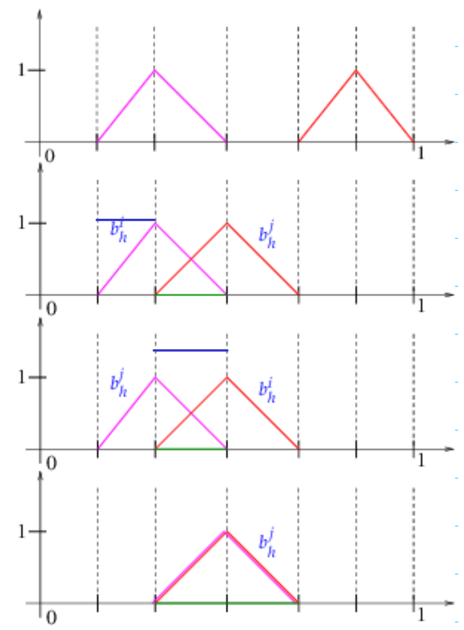
We study its Galerkin discretization based on the trial/test space $\mathcal{S}_{1,0}^0(\mathcal{M})$ on an equidistant mesh of $]0,1[$ with M cells, using the standard tent function basis. Compute the resulting Galerkin matrix.

$$\int_0^1 b_h^j(x)\frac{db_h^i}{dx}(x)\,dx =$$

$$\int_a^b \frac{db_h^j}{dx}(x)\frac{db_h^i}{dx}(x)\,\mathrm{d}x = \begin{cases} 0 & \text{, if } |i-j|\geq 2 \rightarrow \\ * & \\ \cancel{-\frac{1}{h}\cdot\frac{1}{2}h} & \\ -\frac{1}{h_{i+1}} \;/\; -\frac{1}{2} & \text{, if } j = i+1 \rightarrow \\ -\frac{1}{h_i} \;/\; \frac{1}{2} & \text{, if } j = i-1 \rightarrow \\ \frac{1}{h_i}+\frac{1}{h_{i+1}} & \text{, if } 1\leq i=j\leq M-1 \rightarrow \\ \frac{1}{2}-\frac{1}{2}=0 & \end{cases}$$

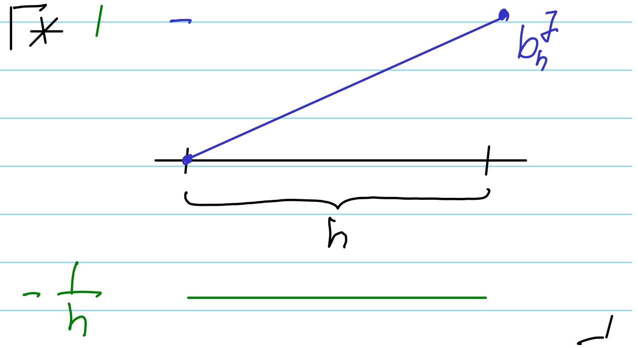
$\Updownarrow$

$-u'' = f$ on $[a,b]$



$\hookrightarrow$ 0 on diagonal of Galerkin matrix

$$\begin{bmatrix} 0 & -\frac{1}{2} & & 0 \\ \frac{1}{2} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{1}{2} \\ 0 & & \frac{1}{2} & 0 \end{bmatrix} \quad \text{on } \mathcal{S}_{1,0}^0(\mathcal{M})$$



③ Q 2.4.6.13F

Explain the mathematics underlying the following code.

C++ code 2.4.5.11: Computation of gradients of barycentric coordinate functions on a triangle ➝ GITLAB

```
Eigen::Matrix<double, 2, 3> gradbarycoordinates(const TriGeo_t& vertices) {
  Eigen::Matrix<double, 3, 3> X;
  // Argument vertices passes the vertex positions of the triangle
  // as the rows cols of a 2×3-matrix, , see
  // Code 2.4.1.3.  The function returns the components of the
  // gradients as the columns of a 2×3-matrix

  // Computation based on (2.4.5.10), solving for the
  // coefficients of the barycentric coordinate functions.
  X.block<3, 1>(0, 0) = Eigen::Vector3d::Ones();
  X.block<3, 2>(0, 1) = vertices.transpose();
  return X.inverse().block<2, 3>(1, 0);
}
```

↳ solve LSE

$$\begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \end{bmatrix} \in \mathbb{R}^{2,3}$$

Barycentric coordinate functions : $\lambda_i(x) = \alpha_i + \beta^i \cdot x$ (affine linear)

for a triangle w/ vertices $a_K^1, a_K^2, a_K^3 \in \mathbb{R}^2$: $\lambda_i(a_K^j) = \delta_{ij}$

Note: $\operatorname{grad} \lambda_i(x) = \beta^i$

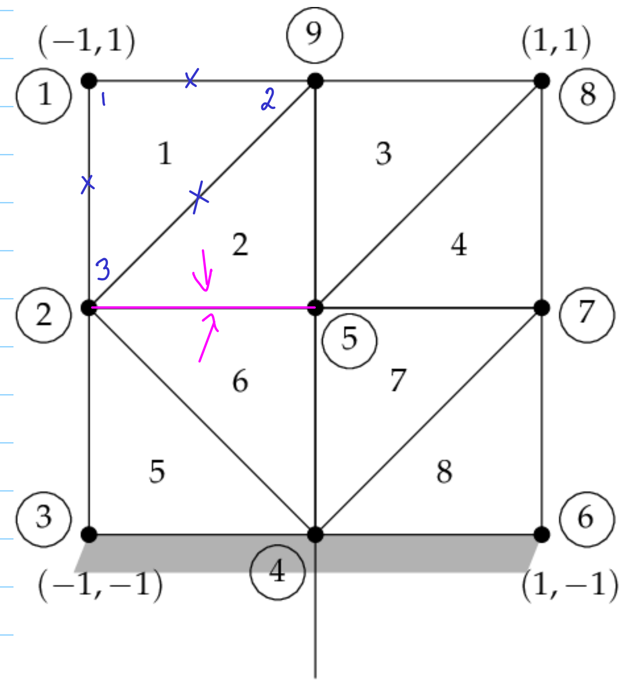
$\downarrow$

LSE

$$\alpha_i + \begin{bmatrix} \beta_1^i & \beta_2^i \end{bmatrix} \cdot \begin{bmatrix} a_1^j \\ a_2^j \end{bmatrix} = \delta_{ij}$$

$$\Leftrightarrow \begin{bmatrix} 1 & a_1^1 & a_2^1 \\ 1 & a_1^2 & a_2^2 \\ 1 & a_1^3 & a_2^3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1^1 & \beta_1^2 & \beta_1^3 \\ \beta_2^1 & \beta_2^2 & \beta_2^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (2.4.5.10)$$

Q 2.4.6.13J :



$\in \mathbb{R}^{9,9}$, $9 = \dim S_1^0(\mathcal{M})$

$\downarrow$

Compute the Galerkin matrix for the bilinear form

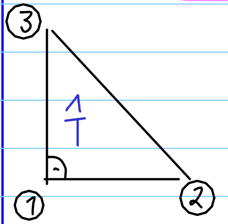
$$a(u, v) = \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x ,$$

$u, v \in H^1(\Omega)$, $\Omega =]-1,1[^2$, and the trial/test space $\mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ shown in Fig. 85, using the tent function bases numbered according to the numbering of the nodes of the mesh as indicated in Fig. 85.

Hint. All triangles of the mesh are congruent. You may also use the formula

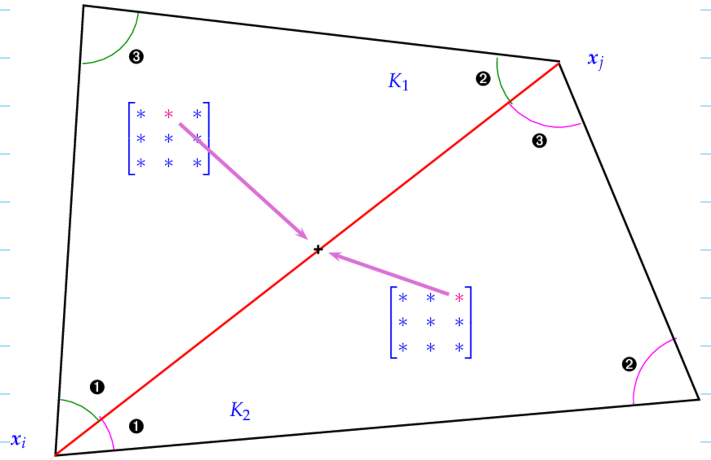
$$\mathbf{A}_K = \frac{1}{2}\begin{bmatrix} \cot\omega_3 + \cot\omega_2 & -\cot\omega_3 & -\cot\omega_2 \\ -\cot\omega_3 & \cot\omega_3 + \cot\omega_1 & -\cot\omega_1 \\ -\cot\omega_2 & -\cot\omega_1 & \cot\omega_2 + \cot\omega_1 \end{bmatrix}, \qquad (2.4.5.8)$$

giving the $\mathcal{S}_1^0(\mathcal{M})$-element matrix for $-\Delta$ on a triangle with angles ω_i, $i = 1, 2, 3$.



Assembly : ("edge oriented")

$$A_K = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$



$(\mathbf{A})_{ij}$ by summing entries of two element matrices

4

$$
\begin{array}{c|ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & 1^* & -\tfrac12 & & & & & & & -\tfrac12 \\
2 & -\tfrac12 & 2^* & -\tfrac12 & 0 & -1 & & & & 0 \\
3 & & & & & & & & & \\
4 & & & & & & & & & \\
5 & & & & & & & & & \\
6 & & & & & & & & & \\
7 & & & & & & & & & \\
8 & & & & & & & & & \\
9 & -\tfrac12 & & & & & & & &
\end{array}
$$

*Trick: $a(\{x \to 1\}, v) = 0$

$\Rightarrow$ Row/column sums of Gal. Mat. $= 0$

$\sum_i b_h^i = 1 \quad , \{b_h^i\} \hat{=}$ tent function basis of $S_1^0(\mathcal{M})$

$$0 = a\Big(\underbrace{\sum_j b_h^j}_{=1}, b_h^i\Big) = \sum_j (A)_{ij}$$

Q 2.4.6.13 K

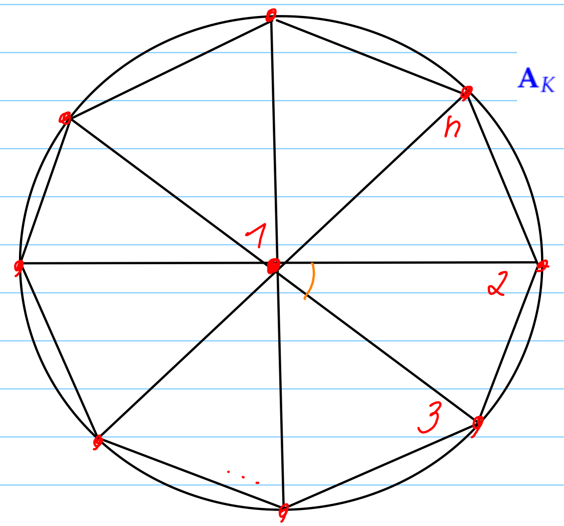
The cells of a triangular mesh $\mathcal{M}$ are generated by connecting all n corners of a regular polygon with diameter 2 with its center. On this mesh we consider the finite element space $\mathcal{S}_1^0(\mathcal{M})$ and the corresponding Galerkin matrix for the bilinear form

$$\mathsf{a}(u,v) = \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, \mathrm{d}\boldsymbol{x} \qquad u,v \in H^1(\Omega)\,,$$

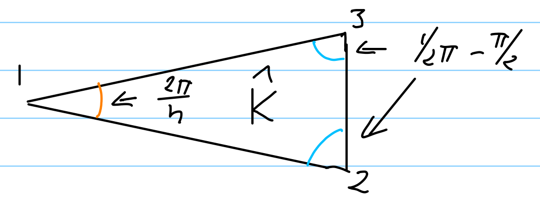
where Ω is the interior of the polygon. We assume that the standard tent function basis of $\mathcal{S}_1^0(\mathcal{M})$ is used and that their numbering starts with the central node and then continues counterclockwise through the corners of the polygon.

$\dim S_1^0(\mathcal{M}) = n+1$

Hint. The formula (2.4.5.8) can be used.



$$\mathbf{A}_K = \frac12 \begin{bmatrix} \cot\omega_3 + \cot\omega_2 & -\cot\omega_3 & -\cot\omega_2 \\ -\cot\omega_3 & \cot\omega_3 + \cot\omega_1 & -\cot\omega_1 \\ -\cot\omega_2 & -\cot\omega_1 & \cot\omega_2 + \cot\omega_1 \end{bmatrix}. \qquad (2.4.5.8)$$



$$A_K = \begin{bmatrix} 2\alpha & -\alpha & -\alpha \\ -\alpha & \alpha+\beta & -\beta \\ -\alpha & -\beta & \alpha+\beta \end{bmatrix}$$

$$
\begin{array}{c}
1
\end{array}
\begin{bmatrix}
2n\alpha & -2\alpha & -2\alpha & \cdots & & -2\alpha \\
-2\alpha & 2(\alpha+\beta) & -\beta & 0 & 0 & -\beta \\
\vdots & -\beta & \ddots & \ddots & & 0 \\
\vdots & 0 & -\beta & \ddots & \ddots & \vdots \\
\vdots & & 0 & \ddots & \ddots & 0 \\
\vdots & 0 & & \ddots & \ddots & -\beta \\
-2\alpha & -\beta & 0 & 0 & -\beta & 2(\alpha+\beta)
\end{bmatrix}
$$

5

Q1.9.0.13C

Appealing to the following estimate

Theorem 1.9.0.10. Multiplicative trace inequality (MTI)

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) .$$

prove that for the linear variational problem

$$u \in H^1(\Omega): \quad \underbrace{\int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x + \int_{\partial\Omega} uv \,\mathrm{d}S}_{=a(u,v)} = \underbrace{\int_{\partial\Omega} hv \,\mathrm{d}S}_{\ell(v)} \quad \forall v \in H^1(\Omega) , \qquad h \in L^2(\Gamma)$$

both the bilinear form and the right-hand side linear form are continuous on $H^1(\Omega)$.

Tool: MTI & CSI (Cauchy-Schwarz inequality in $L^2(\Omega)/L^2(\Gamma)$)

To show: $|a(u,v)| \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall u,v \in H^1(\Omega)$

$$|a(u,v)| \overset{CSI}{\leq} \|\mathrm{grad}\, u\|_{L^2(\Omega)} \cdot \|\mathrm{grad}\, v\|_{L^2(\Omega)} + \|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)}$$

$$\overset{MTI}{\leq} \|\mathrm{grad}\, u\|_{L^2(\Omega)} \cdot \|\mathrm{grad}\, v\|_{L^2(\Omega)} + \tilde{C}\|u\|^{1/2}_{\cancel{L^2(\Omega)}\, H^1(\Omega)} \|u\|^{1/2}_{H^1(\Omega)} \|v\|^{1/2}_{\cancel{L^2(\Omega)}\, H^1(\Omega)} \|v\|^{1/2}_{H^1(\Omega)}$$

$$\leq \|\mathrm{grad}\, u\|_{L^2(\Omega)} \cdot \|\mathrm{grad}\, v\|_{L^2(\Omega)} + \tilde{C}\|u\|_{H^1(\Omega)} \cdot \|v\|_{H^1(\Omega)}$$

$$\leq \max\{1, \tilde{C}\}\, 2\, \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$$|\ell(v)| \overset{CSI}{\leq} \|h\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \leq \|h\|_{L^2(\Gamma)} \cdot \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$

Q1.9.0.13E

Another stability issue: How does a perturbation (measured in $L^2(\partial\Omega)$-norm) $\delta h \in L^2(\partial\Omega)$ of the Neumann data $h \in L^2(\partial\Omega)$ in the linear variational problem

$$u \in H^1_*(\Omega): \quad \int_\Omega \alpha(x)\, \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x + \int_{\partial\Omega} hv \,\mathrm{d}S \quad \forall v \in H^1_*(\Omega) . \qquad (*)$$

impact the energy norm of the solution of that variational problem?

vanishing mean

uniformly positive definite, here $\alpha = I$

$$H^1_*(\Omega) = \{ v \in H^1(\Omega) : \int_\Omega v(x)\,dx = 0 \}$$

Theorem 1.8.0.20. Second Poincaré-Friedrichs inequality

If $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is bounded and connected, then

$$\exists C = C(\Omega) > 0: \quad \|u\|_0 \leq C\, \mathrm{diam}(\Omega) \|\mathbf{grad}\, u\|_0 \quad \forall u \in H^1_*(\Omega) .$$

$\|u\|_{H^1} \leq C \|\mathrm{grad}\, u\|_{L^2}$

$\Rightarrow$ (*) has unique solution

Perturbation $\quad h \longmapsto h + \delta h$

$\qquad\qquad\quad u \longmapsto u + \delta u$

By linearity: $\quad \int_\Omega \mathrm{grad}\, \delta u \cdot \mathrm{grad}\, v \,dx = \int_{\partial\Omega} \delta h \cdot v \,dS$

$v := \delta u \in H^1_*(\Omega)$ $\triangleright$ $\tilde{C}\|\delta u\|^2_{H^1} \leq \|\mathrm{grad}\, \delta u\|^2 = \|\delta u\|^2_a \overset{C.S.I.}{\leq} \|\delta h\|_{L^2(\Gamma)} \|\delta u\|_{L^2(\Gamma)}$

$$\overset{MTI}{\leq} C\|\delta h\|_{L^2(\Gamma)} \cancel{\|\delta u\|_{H^1(\Omega)}}$$

$$\triangleright \quad \tilde{C}\|\delta u\|_{H^1(\Omega)} \leq C\|\delta h\|_{L^2(\Gamma)}$$

6

Q 1.9.0.13 F

For a bounded polygon $\Omega \subset \mathbb{R}^2$, we consider the linear variational problem

$$u \in H^1(\Omega): \quad \int_\Omega e^{\|x\|} \operatorname{\mathbf{grad}} u(x) \cdot \operatorname{grad} v(x)\, dx = \int_{\partial\Omega} \underbrace{x_1}_{\hat{=} h} v(x)\, dx, \quad \forall v \in H^1(\Omega). \tag{1.9.0.14}$$

State the boundary value problem satisfied by sufficiently smooth solutions of Eq. (1.9.0.14). Write down a domain Ω for which Eq. (1.9.0.14) has a solution.

BVP :
$$\begin{cases} -\operatorname{div}(e^{\|x\|} \operatorname{grad} v) = 0 & \text{in } \Omega \\ e^{\|x\|} \operatorname{grad} u(x) \cdot n(x) = x_1 & x \in \partial\Omega \end{cases}$$

Sect 1.8 :

$$u \in H^1(\Omega): \quad \int_\Omega \kappa(x) \operatorname{\mathbf{grad}} u \cdot \operatorname{\mathbf{grad}} v\, dx - \int_{\partial\Omega} h v\, dS = \int_\Omega f v\, dx \quad \forall v \in H^1(\Omega). \tag{1.8.0.12}$$

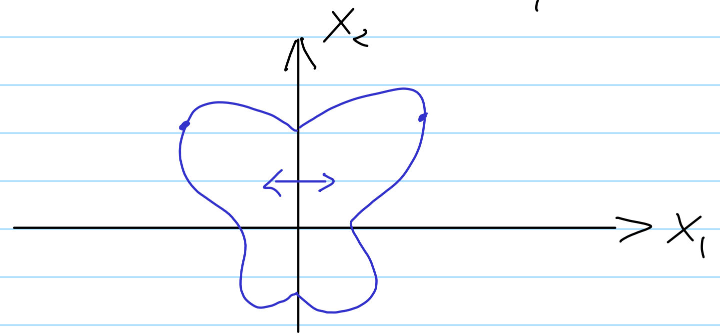
Observation: when we test (1.8.0.12) with $v \equiv 1$ ▶ $\boxed{-\int_{\partial\Omega} h\, dS = \int_\Omega f\, dx}$ (1.8.0.13)

$$\downarrow$$

Necessary condition : $\int_{\partial\Omega} x_1\, dS(x) = 0$

$\downarrow$

satisfied for all Ω symmetric to x_2-axis



1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session March 22

HW 2-1 d)

LVP : $u \in V_0$: $a(u,v) = \ell(v) \quad \forall v \in V_0$ (1)

a: s.p.d., ℓ: cont. w.r.t. $\|\cdot\|_a$

+ Galerkin discretization w/ $V_{0,h} \subset V_0$, $\dim V_{0,h} < \infty$

$u_h \in V_{0,h}$: $a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_{0,h}$ (2)

[$v \to v_h$ in (1)] : $a(u - u_h, v_h) = 0 \quad \forall v_h \in V_{0,h}$

$\hat{=}$ Galerkin orthogonality (GO)

Associated QMP :

$u_h := \underset{v_h \in V_{0,h}}{\operatorname{argmin}} J(v_h)$: $J(v_h) = \frac{1}{2} a(v_h, v_h) - \ell(v_h)$

[Needed: a is bilinear, ℓ linear]

$J(u_h) - J(u) =$

$= \frac{1}{2} a(u_h, u_h) - \ell(u_h) - \frac{1}{2} a(u,u) + \ell(u)$

$\overset{(2)\&(1)}{=} \frac{1}{2} \ell(u_h) - \ell(u_h) - \frac{1}{2} \ell(u) + \ell(u)$

$= \frac{1}{2} \ell(u - u_h)$

$\overset{(1)}{=} \frac{1}{2} a(u, u - u_h)$

$\overset{GO}{=} \frac{1}{2} a(u - u_h, u - u_h) \overset{\text{def. of } \|\cdot\|_a}{=} \frac{1}{2} \|u - u_h\|_a^2$

[$a(u-u_h, u-u_h) = a(u, u-u_h) - \underbrace{a(u_h, u-u_h)}_{=0 \text{ by GO}}$]

Q 2.6.2.13.C : $\hat{=}\, A \in \mathbb{R}^{N,N}$

For a triangular mesh $\mathcal{M}$ with $\sharp\mathcal{V}(\mathcal{M})$ vertices, $\sharp\mathcal{E}(\mathcal{M})$ edges, and $\sharp\mathcal{M}$ cells give sharp upper bounds for the number of non-zero entries of the Galerkin matrix arising from the finite element discretization of

$$u \in H^1(\Omega): \int_\Omega \alpha(\mathbf{x})\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v + c(\mathbf{x})u\,v\,\mathrm{d}\mathbf{x} = \int_\Omega f v\,\mathrm{d}\mathbf{x} \quad \forall v \in H^1(\Omega)\,, \tag{2.1.2.2}$$

with trial and test space $\mathcal{S}^0_p(\mathcal{M})$, $p = 1,2$.

generic coefficient

$p = 1$: $\mathrm{nnz}(A) = 2\,\#\mathcal{E}(\mathcal{M}) + \#\mathcal{V}(\mathcal{M})$

off-diagonal entries

$\hat{=}\, N$: no. of diagonal entries

cf. Sect. 2.4.

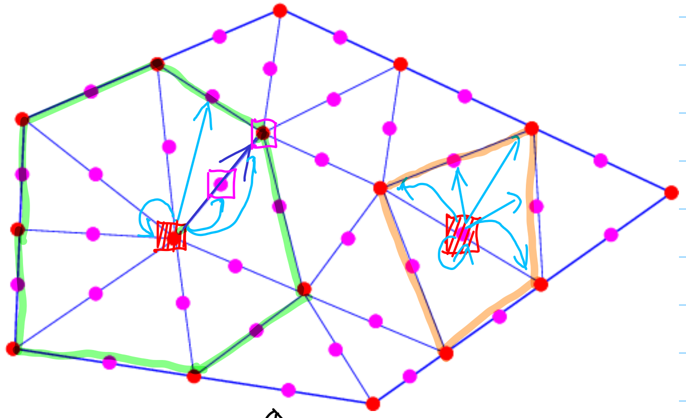
$$\left\{ \begin{array}{c} \text{Nodes } \mathbf{x}_i, \mathbf{x}_j \in \mathcal{V}(\mathcal{M}) \\ \text{not connected by an edge} \end{array} \right. \Leftrightarrow \boxed{\mathrm{Vol}(\mathrm{supp}(b_h^i) \cap \mathrm{supp}(b_h^j)) = 0} \Rightarrow (\mathbf{A})_{ij} = 0\,.$$

$p = 2$: $V_{0,h} = \mathcal{S}^0_2(\mathcal{M})$

We rely on a set $\mathcal{N}$ of interpolation nodes

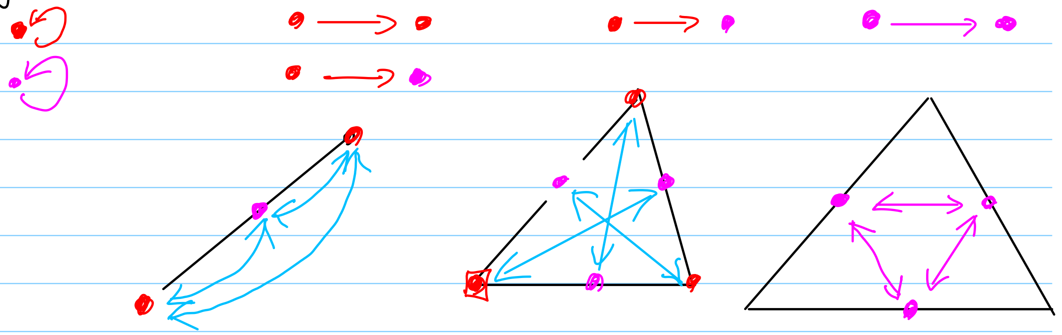
$$\mathcal{N} := \mathcal{V}(\mathcal{M}) \cup \{\text{midpoints of edges}\}\,,$$
$$\mathcal{N} = \{\mathbf{p}_1, \ldots, \mathbf{p}_N\} \quad \text{(ordered)}\,.$$

$N = \#\mathcal{V} + \#\mathcal{E}$



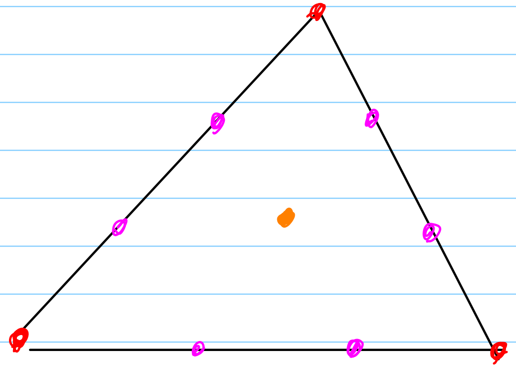
Locations of GSF for $\mathcal{S}^0_2(\mathcal{M})$

$$\mathrm{nnz}(A) = \underbrace{\#\mathcal{V} + \#\mathcal{E}}_{\hat{=}\ \text{diagonal entries}} + \underbrace{2\cdot 3\,\#\mathcal{E}} + 2\cdot 3\,\#\mathcal{M} + 2\cdot 3\,\#\mathcal{M}$$



$$= \#\mathcal{V} + 7\cdot\#\mathcal{E} + 12\,\#\mathcal{M}$$

$p = 3$



③

Simon Bolt to Everyone 05:16 PM

In the context of FEM, we have only seen identical trial/test spaces so far. So when might different trial/test spaces be useful?

rarely used : Petrov-Galerkin method

mainly for non-symmetric problems

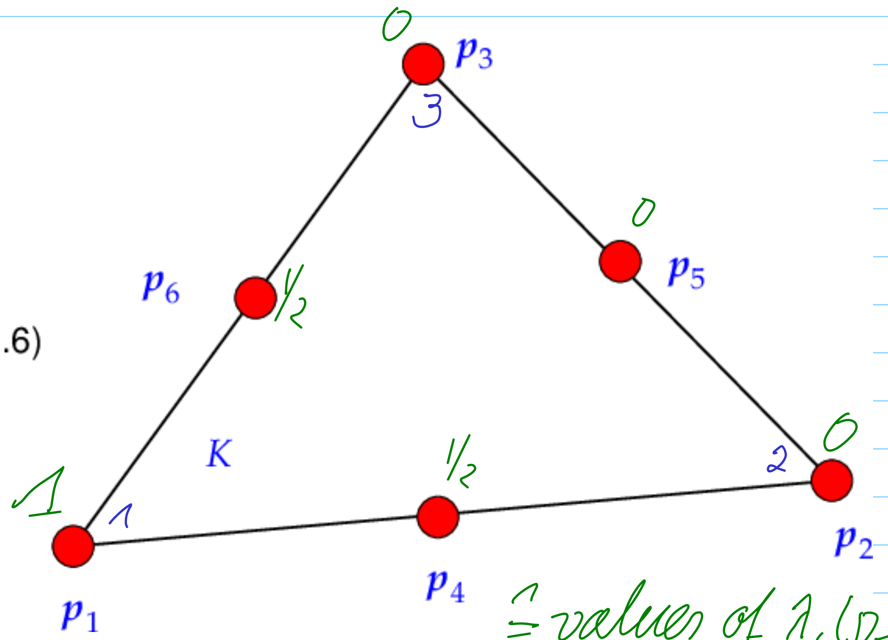
R.Q : 2.6.2.13.F

Express the local shape functions for linear Lagrangian finite elements on a triangle as linear combinations of the quadratic local shape functions as given in

$$b_K^1 = (2\lambda_1 - 1)\lambda_1 , \quad b_K^2 = (2\lambda_2 - 1)\lambda_2 ,$$
$$b_K^3 = (2\lambda_3 - 1)\lambda_3 , \quad b_K^4 = 4\lambda_1\lambda_2 , \qquad (2.6.1.6)$$
$$b_K^5 = 4\lambda_2\lambda_3 , \quad b_K^6 = 4\lambda_1\lambda_3 .$$

Local spaces: $\underset{\mathrm{Span}\{\lambda_1,\lambda_2,\lambda_3\}}{\overset{}{\mathcal{S}_1^0(K)}} = \mathcal{P}_1(\mathbb{R}^2) \subset \mathcal{P}_2(\mathbb{R}^2) = \underset{\mathrm{Span}\{b_K^1,\dots,b_K^6\}}{\mathcal{S}_2^0(K)}$

$$b_K^1 = (2\lambda_1 - 1)\lambda_1 ,$$
$$b_K^2 = (2\lambda_2 - 1)\lambda_2 ,$$
$$b_K^3 = (2\lambda_3 - 1)\lambda_3 ,$$
$$b_K^4 = 4\lambda_1\lambda_2 , \qquad (2.6.1.6)$$
$$b_K^5 = 4\lambda_2\lambda_3 ,$$
$$b_K^6 = 4\lambda_1\lambda_3 .$$



≙ values of $\lambda_1(p_j)$

$$b_K^\ell(p_j) = \delta_{ij}$$

$$\lambda_k = \sum_{j=1}^{6} \lambda_k(p_j)\, b_K^j$$

e.g.: $\lambda_1 = b_K^1 + \frac{1}{2} b_K^4 + \frac{1}{2} b_K^6$

④

Characterize the space of gradients of $\mathcal{P}_p(\mathbb{R}^2)$ and $\mathcal{Q}_p(\mathbb{R}^2)$ as spaces of componentwise polynomial vectorfields.

- $\operatorname{grad} \mathcal{P}_p(\mathbb{R}^2) \subset \mathcal{P}_{p-1}(\mathbb{R}^2) \times \mathcal{P}_{p-1}(\mathbb{R}^2)$

$$f(\underline{x}) = \sum_{0 \le i+j \le p} \alpha_{ij}\, x_1^i x_2^j \in \mathcal{P}_p(\mathbb{R}^2)$$

$$\operatorname{grad} f(\underline{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \sum_{0\le i+j\le p} \alpha_{ij}\, i x_1^{i-1} x_2^j \\ \sum_{0\le i+j\le p} \alpha_{ij}\, j x_1^{i} x_2^{j-1} \end{bmatrix} \begin{matrix} \in \mathcal{P}_{p-1} \\ \in \mathcal{P}_{p-1} \end{matrix}$$

$$\in \mathcal{P}_{p-1}(\mathbb{R}^2) \times \mathcal{P}_{p-1}(\mathbb{R}^2)$$

Supplement:

$$\operatorname{grad} \mathcal{P}_p(\mathbb{R}^3) = \left\{ \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in \mathcal{P}_{p-1}(\mathbb{R}^2) \times \mathcal{P}_{p-1}(\mathbb{R}^2) : \frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} \right\}$$

- $\operatorname{grad} \mathcal{Q}_p(\mathbb{R}^2) = \left(\mathcal{P}_{p-1}(\mathbb{R}) \otimes \mathcal{P}_p(\mathbb{R})\right) \times \left(\mathcal{P}_p(\mathbb{R}) \otimes \mathcal{P}_{p-1}(\mathbb{R})\right)$

$$f(\underline{x}) = \sum_{i=0}^{p} \sum_{j=0}^{p} \alpha_{ij}\, x_1^i x_2^j$$

$$\operatorname{grad} f(\underline{x}) = \begin{bmatrix} \sum_{i=0}^{p-1} \sum_{j=0}^{p} \beta_{ij}\, x_1^i x_2^j \\ \sum_{i=0}^{p} \sum_{j=0}^{p-1} \gamma_{ij}\, x_1^i x_2^j \end{bmatrix} \begin{matrix} \in \mathcal{P}_{p-1}(\mathbb{R}) \otimes \mathcal{P}_p(\mathbb{R}) \\ \in \mathcal{P}_p(\mathbb{R}) \otimes \mathcal{P}_{p-1}(\mathbb{R}) \end{matrix}$$

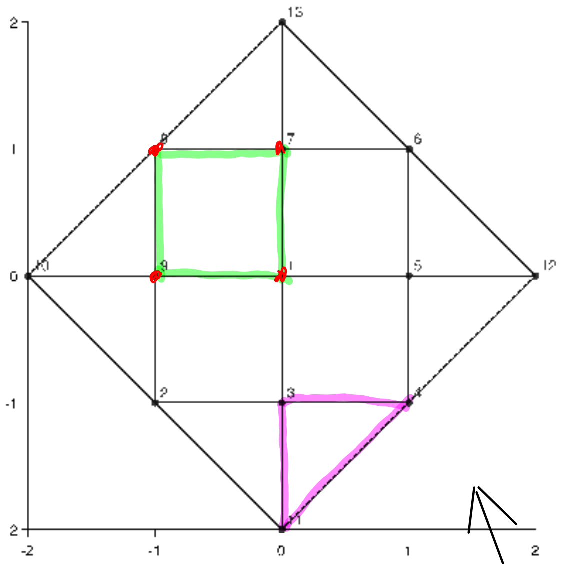
Q 2.6.2.13.H

We consider the bilinear form

$$\mathrm{b}(u,v) := \int_{\partial\Omega} (1 + \|\boldsymbol{x}\|)\, u(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}S\,, \tag{2.6.2.14}$$

treat as "generic coefficient"

where $\partial\Omega$ is the boundary of the domain sketched in Fig. 121.



Writing $\mathcal{M}$ for the mesh drawn in Fig. 121, what is the maximal number of nonzero entries of the Galerkin matrix arising from the finite element Galerkin discretization of $\mathrm{b}(\cdot,\cdot)$ using $\mathcal{S}_1^0(\mathcal{M})$ equipped with the standard nodal basis as trial and test space?

▷ Q 2.6.2.13.C

Q 2.6.2.13.J

We perform the Galerkin discretization of

$$\mathrm{a}(u,v) := \int_{\partial\Omega} u(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}S\,,$$

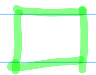
where $\partial\Omega$ is the boundary of the domain from Fig. 121, by means of the finite element space $\mathcal{S}_1^0(\mathcal{M})$ equipped with the standard nodal basis. Here $\mathcal{M}$ is the hybrid mesh sketched in Fig. 121.

Compute the element matrices for the two cells of the mesh of Fig. 121 whose vertices are the nodes with the following numbers:

(i) nodes 9, 1, 7, 8 (quadrilateral),

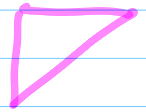
(ii) nodes 3, 11, 4 (triangle)

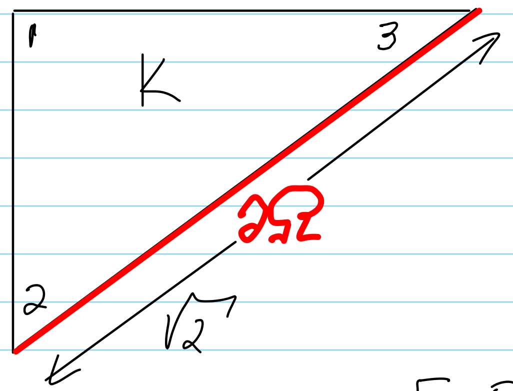
Any local numbering of the nodes can be used.

 : $A_{\square} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \vdots \\ 0 & & \ddots & \\ 0 & \cdots & & 0 \end{bmatrix} \in \mathbb{R}^{4,4}$

An element matrix

$\square$ does not have an edge on $\partial\Omega$





← local numbering

LSF $\hat{=}$ barycentric coordinate functions

$$A_{\triangleright} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2}/3 & \sqrt{2}/6 \\ 0 & \sqrt{2}/6 & \sqrt{2}/3 \end{bmatrix} \in \mathbb{R}^{3,3}$$

$$a(u,v) = \int_{\partial\Omega} u\, v \, dS$$

$$a_K(\lambda_2, \lambda_3) = \int \lambda_2(x)\, \lambda_3(x) \, dS(x) = \sqrt{2}/6$$

Lemma 2.7.5.5. Integration of powers of barycentric coordinate functions

For any non-degenerate d-simplex K with barycentric coordinate functions $\lambda_1, \dots, \lambda_{d+1}$ and exponents $\alpha_j \in \mathbb{N}$, $j = 1, \dots, d+1$,

$$\int_K \lambda_1^{\alpha_1} \cdot \dots \cdot \lambda_{d+1}^{\alpha_{d+1}} \, \mathrm{d}\boldsymbol{x} = d!|K| \frac{\alpha_1!\alpha_2! \cdot \dots \cdot \alpha_{d+1}!}{(\alpha_1 + \alpha_2 + \dots + \alpha_{d+1} + d)!} \quad \forall \boldsymbol{\alpha} \in \mathbb{N}_0^{d+1}. \tag{2.7.5.6}$$

▷ Apply w/ $d = 1$ (edge!)

$|K| = \sqrt{2}$

1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session March 29

RQ 2.7.5.44 B

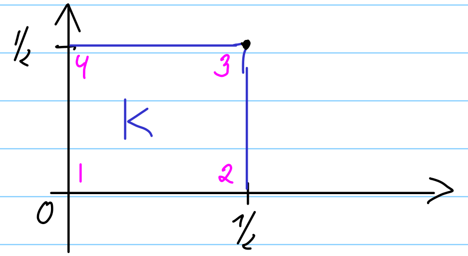
Let $b_K^1, \dots, b_K^4$ be the local shape functions for the finite element space $\mathcal{S}_1^0(\mathcal{M})$ and a rectangle K with vertices

$$a_K^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad a_K^2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad a_K^3 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad a_K^4 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}.$$

What is the value of

$$\int_K (b_K^1(\mathbf{x}))^p (b_K^3(\mathbf{x}))^q \, d\mathbf{x}, \quad p, \in \mathbb{N}_0 \, ?$$

local shape functions $\in Q_1(\mathbb{R}^2)$



$$b_K^1(x) = (1-2x_1)(1-2x_2)$$

$$b_K^3(x) = 4x_1x_2$$

$$\int_K b_K^1(x)^p \, b_K^3(x)^q \, dx = \int_0^{1/2}\int_0^{1/2} (1-2x_1)^p(1-2x_2)^p(2x_1)^q(2x_2)^q \, dx_1 dx_2$$

$$= \int_0^{1/2} (1-2x_1)^p(2x_1)^q \, dx_1 \cdot \int_0^{1/2} (1-2x_2)^p(2x_2)^q \, dx_2 = I^2$$

Idea: Separation of directions

$$\int_0^{1/2} (1-2t)^p(2t)^q \, dt = \frac{1}{2}\int_0^1 (1-s)^p s^q \, ds = I$$

cf. Proof of Lemma 2.7.5.5 : $I = B(p+1, q+1)$

↳ Euler Beta Function

② $B(p+1, q+1) = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} = \frac{p!\,q!}{(p+q+1)!} = I$

Ad LSF:

Defining property: $b_K^j \in \mathcal{Q}_1(\mathbb{R}^2)$

$b_K^j(a_K^\ell) = \delta_{k,\ell} \quad , k,\ell = 1,\dots,4$

↳ local cardinal basis property

HW 2-9) d)

In the file `lfppdofhandling.cc` complete the C++ function

```
std::array<std::size_t,3> countEntityDofs(const
    lf::assemble::DofHandler &dofhandler);
```

that returns the number of global shape functions (managed by the local→global index mapping encoded in `dofhandler`) *associated with* mesh entities of co-dimension 0, 1, and 2 respectively (the co-dimension also serves as index for the returned array).

In LF++: no by calling NumInteriorDofs()

C++11 code 2.9.2: Sub-problem (2-9.d): Implementation of `countEntityDofs()` ➙ GITHUB

```
std::array<std::size_t, 3>
countEntityDofs(const lf::assemble::DofHandler &dofhandler) {
  std::array<std::size_t, 3> entityDofs;
  // Idea: iterate over entities in the mesh and get interior number of
      dofs for
  // each
  std::shared_ptr<const lf::mesh::Mesh> mesh = dofhandler.Mesh();
  for (std::size_t codim = 0; codim <= 2; ++codim) {
    entityDofs[codim] = 0;
    for (const auto *el : mesh->Entities(codim)) {
      if (el->RefEl() == lf::base::RefEl::kQuad()) {
        throw "Only triangular meshes are allowed!";
      }
      entityDofs[codim] += dofhandler.NumInteriorDofs(*el);
    }
  }
  return entityDofs;
}
```

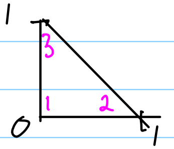
◁ Not needed

HW 2-9 f)

K triangle w/ barycentric coordinate functions λ_i, $i = 1,2,3$

$\int_K \lambda_\ell(\underline{x})\,dx = ?$

I. By affine transformation: [→ §2.7.5.12]

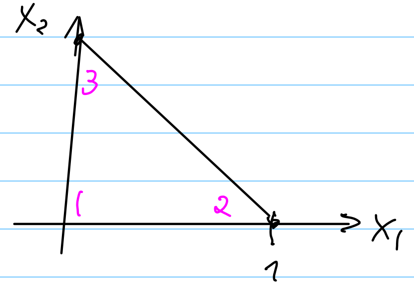
$\Phi_K : \hat{K} =$  $\longrightarrow K$, $\hat{\lambda}_i = \Phi_K^* \lambda_i$

[Lemma 2.8.1.4, §2.8.1.5]

③

$$\int_K \lambda_i(x)\,dx = \int_{\hat K} \hat\lambda_i(\hat x)\,\underbrace{|\det D\Phi_K(\hat x)|}_{\text{constant } = 2|K|}\,d\hat x$$

$$\int_{\hat K} \hat\lambda_2(\hat x)\,d\hat x = \int_0^1\int_0^{1-\hat x_1} \hat x_1\,d\hat x_1$$



$$= \int_0^1 (1-\hat x_1)\,\hat x_1\,d\hat x_1 = 1/6$$

$$\int_K \lambda_2(x)\,dx = 1/3\,|K|$$

II.

Lemma 2.7.5.5. Integration of powers of barycentric coordinate functions

For any non-degenerate d*-simplex* K *with barycentric coordinate functions* $\lambda_1,\dots,\lambda_{d+1}$ *and exponents* $\alpha_j \in \mathbb{N}$, $j = 1,\dots,d+1$,

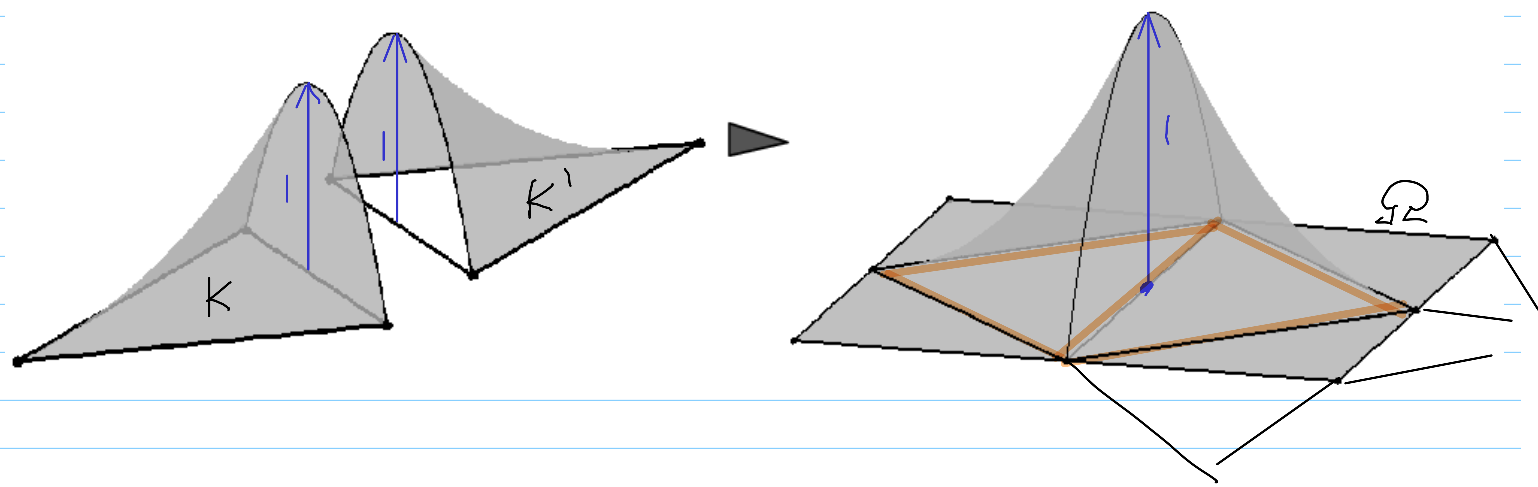
$$\int_K \lambda_1^{\alpha_1}\cdot\dots\cdot\lambda_{d+1}^{\alpha_{d+1}}\,\mathrm{d}x = d!|K|\,\frac{\alpha_1!\alpha_2!\cdot\dots\cdot\alpha_{d+1}!}{(\alpha_1+\alpha_2+\dots+\alpha_{d+1}+d)!}\quad \forall\alpha\in\mathbb{N}_0^{d+1}\,. \tag{2.7.5.6}$$

Apply: $d=2$, $\alpha_i = 1$, $\alpha_k = 0$ for $k \neq i$

$$\int_K \lambda_i(x)\,dx = 2!\,|K|\,\frac{1!}{(2+1)!} = 1/3\,|K|$$

Local shape function $\longleftrightarrow$ global shape functions

For $\mathcal{S}_2^0(\mathcal{M})$:



6

1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session
April 12, 2021

Q1:

Big Picture
by Nico Graf - Monday, 12 April 2021, 12:38 AM

In the last Q&A you gave us the hint to focus on the big picture when studying for the midterms and in general reviewing the course material.
So it would be interesting to hear what you regard as the most valuable connections between the topics we look at so far.

As far as I currently understand the material I would summarize like this:

First **Problem Definition**:
- We get a minimization problem (i.e. an operator on a function that we want to minimize)
- Then transform it into a **Linear Variational Problem**
  -> We can check if there are solutions by checking the **necessary conditions** (e.g. continuity of the Linear and bilinear form)
    -> for this we can use theorems like **Cauchy Schwarz**, **Mulitplicative Trace inequality** or **Poncre Friedierichs Theorems**

Second **Modelling**
- Once we have checked the existance of a solution we can use **Finite Element methods** to find a good approximation of the solution
  -> The essence here is that we are given a **mesh** and a **degree of Freedom** which then gives us the framework for which **basis functions** we can use,
    -> in particular the number of d.o.f 's per cell and the shape of the cell will dictate what basis functions and what degree we use.
  -> given the basis functions we can then build the **galerkin matrix** and the **righthandside Vector**
  -> then we just need to solve the given system for the coefficients correspoinding to the choosen basis

Third **Refinement and Assessment**
- When we compute a solution we then want to know how large the error is since with FEM we are only able to compute an approximation.
  -> We are interested in how fast the error converges and find this out by studying the erors for different mesh refinements or increase of d.o.f
    -> typical categories are **algebraic convergence** and **exponetial convergence**
      -> for algebraic convergence we can then as well determine the **rate of the convergence**
    -> In order to estimate the error and the convergence we have theorems like (3.3.5.6 **Best apporximation error estimates** ...)
      -> Those esimated tell us that the convergence heavily depends on the **"degree* of the Sobolev space** in which the function is.

I was wondering if I'm missing some important Keyword in my very short summary or if you have some additional remarks ?
Thanks in advance.

2

Chapter 1 :

First **Problem Definition**:
- We get a minimization problem (i.e. an operator on a function that we want to minimize)
- Then transform it into a **Linear Variational Problem**
  -> We can check if there are solutions by checking the **necessary conditions** (e.g. continuity of the Linear and bilinear form)
    -> for this we can use theorems like **Cauchy Schwarz**, **Mulitplicative Trace inequality** or **Poncre Friedierichs Theorems**

Quadratic minimization problem $\Longleftrightarrow$ LVP *

$J(v) := \frac{1}{2}a(v,v) - \ell(v) \to \min$

Energy norm $\|\cdot\|_a$ continuity w.r.t. $\|\cdot\|_a$

* set function spaces : Sobolev space

2nd-elliptic LVP $\Longleftrightarrow$ BVP (PDE + b.c.*)

Tool : I.p.b.

* natural & essential
* admissible data

Why do we need BVP : - to construct manufactured solutions
- to understand smoothness
[- for duality estimates]

Chapter 2 : Discretization

~~Second Modelling~~
- Once we have checked the existance of a solution we can use **Finite Element methods*** to find a (good) approximation of the solution
  -> The essence here is that we are given **a mesh** and a **degree of Freedom** which then gives us the framework for which **basis functions** we can use,
    -> in particular the number of d.o.f 's per cell and the shape of the cell will dictate what basis functions and what degree we use.
  -> given the basis functions we can then build the **galerkin matrix** and the **righthandside Vector**
  -> then we just need to solve the given system for the coefficients correspoinding to the choosen basis

* Galerkin discretization , FEM as special case

FEM : 1) mesh $\mathcal{M}$
2) $\mathcal{M}$-p.w. polynomial trial / test space
3) locally supported basis functions* associated with mesh entities.

* GSF $\longleftrightarrow$ d.o.f. (basis exp. coefficients)

$\Updownarrow$

LSF

Focus on Lagrangian FE spaces

LVP $\xRightarrow{(FEM)}$ LSE $A\vec{\mu} = \vec{\varphi}$

- Assembly via element matrices & vectors and local $\to$ global index mappings.

- Parametric FEM

Chapter: Convergence

Third **Refinement and Assessment**

- When we compute a solution we then want to know how large ~~the~~ a norm of error is since with FEM we are only able to compute an approximation.
  - -> We are interested in how fast the error converges and find this out by studying the erors for different mesh refinements or increase of d.o.f
    - -> typical categories are **algebraic convergence** and **exponetial convergence**
      - -> for algebraic convergence we can then as well determine the **rate of the convergence**
    - -> In order to estimate the error and the convergence we have theorems like (3.3.5.6 **Best apporximation error estimates** ...)
      - -> Those esimated tell us that the convergence heavily depends on the **"degree" of the Sobolev space** in which the function is.

All we can get: asymptotic cvg.*

- quasi-optimality of Galerkin solution

depends on FE space & smoothness u

$$* \qquad \|u-u_h\| \leq C h_m^q \,, \quad q \in \mathbb{N}$$

unknown

Still allows to predict expected effort for prescribed accuracy

Appl.: Debugging

- Variational crimes

Paolo Bottoni to Everyone 04:56 PM

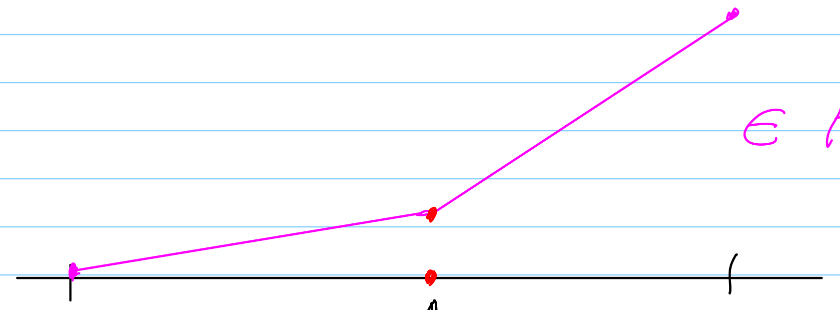
PB

I have a question regarding chapter 1 I couldn't find it earlier but now I did: in remark 1.5.3.10 we have a diagram which tells us that we can only go from LVP to BVP and not the other way. am I reading this diagram wrong?

No equivalence : LVP $\not\Leftrightarrow$ BVP

1D example w/ discontinuous coeff. α

$$-(\alpha(x)u')' = f \quad \text{in }]a,b[$$



$\in H^1(]a,b[)$

derivative of u here?

4

RQ 3.3.5.9.D

If $\mathcal{M}'$ has been created by regular refinement of a triangular mesh $\mathcal{M}$, how are shape-regularity measures $\rho_{\mathcal{M}}$ and $\rho_{\mathcal{M}'}$ related?

Definition 3.3.2.20. Shape regularity measure

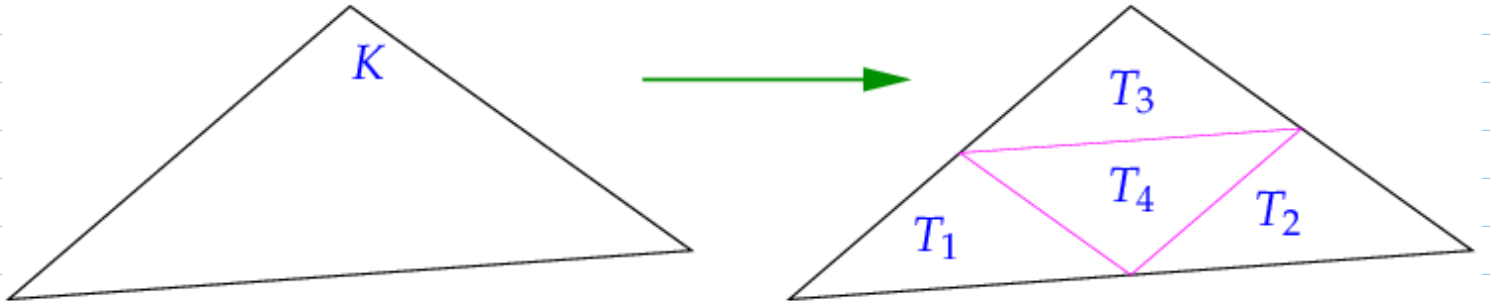
For a simplex $K \in \mathbb{R}^d$ we define its **shape regularity measure** as the ratio

$$\rho_K := h_K^d : |K| \,, \quad h_K := \mathrm{diam}(K)\,,$$

and the shape regularity measure of a simplicial mesh $\mathcal{M} = \{K\}$ as

$$\rho_{\mathcal{M}} := \max_{K \in \mathcal{M}} \rho_K\,.$$

Regular refinement of a triangle



Regular refinement of triangle K into four congruent triangles T_1, T_2, T_3, T_4

K and T_i are similar (same angles!)

$$\rho_K = \rho_{T_i}$$

because $h \to h/2$, $|K| \to \frac{1}{4}\cdot|K|$ here $d = 2$

HW 2-8 g)

- Assembly of r.h.s. vector for $v \mapsto \int_\Omega f(x)v(x)\,dx$ and $\mathcal{S}_1^0(\mathcal{M})$ *

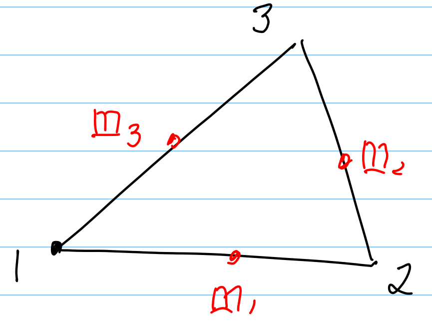
+ Local edge-midpoint quadrature rule for cell K

$$\int_K \varphi(x)\,dx \approx \frac{|K|}{\#\mathcal{E}(K)} \sum_{\ell=1}^{\#\mathcal{E}(K)} \varphi(m_\ell)$$

edge midpoint

* K triangle : LSF are barycentric coord. functions

$$(\vec{\varphi}_K)_1 = \int_K f(x)\lambda_1(x) \approx \frac{|K|}{3}\,\frac{1}{2}\,(f(m_1) + f(m_3))$$



$$\lambda_1(m_1) = 1/2$$
$$\lambda_1(m_2) = 0$$
$$\lambda_1(m_3) = 1/2$$

5

Where is the error

Pseudocode 2.7.4.21: Generic assembly algorithm for finite element right hand side vectors

```
Vector ← assembleRhsVector(Mesh 𝓜) {
  phi = Vector(N); // Allocate zero vector of appropriate length
  foreach K ∈ 𝓜 { // loop over all cells
    // Obtain number Q(K) of local shape functions, see Def. 2.7.4.5
    Qk  = no_loc_shape_functions(K);
    // Local operation: compute element vector, length Qk →
       Def. 2.7.4.5,
    // (usually incurs cost of only "O(1)")
    phi_k = getElementVector(K);
    // Get vector of global indices (length Q(K));
    // Usage of locglobmap as in Ex. 2.7.4.10
    Vector idx = {locglobmap(K,1)),...,locglobmap(K,Qk)};
    // Add local contributions to global right-hand-side vector
    for i:=1 to Qk {
      phi(idx(i)) += phi(idx(i)) + phi_k(i);
    }
  }
  return(phi);
}
```

wrong

Confusing notation:

element vector

$$(\vec{\varphi})_j = \sum_{l=1}^{N_j} \ell_{K_l}(b_h^j{}_{|K_l}) = \sum_{l=1}^{N_j} (\vec{\varphi}_K)_{i(l,j)}\,, \qquad (2.4.6.6)$$

$[\,l \neq I\,]$

$i(l,j) \in \{1,2,3\}$ ≙ index of LSF that is the restriction of GSF #j to K_l

provided by DofHandler

6

RQ 3.3.5.29 G

Appealing to Ex. 1.2.3.45 explain why for a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ and $\mathsf{I}_1 : C^0(\overline{\Omega}) \to \mathcal{S}_1^0(\mathcal{M})$ denoting the nodal interpolation operator according to Def. 3.3.2.1 the *interpolation* error estimate

$$\|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \leq C h_{\mathcal{M}} \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) ,$$

with $C > 0$ depending only on the shape regularity measure $\rho_{\mathcal{M}}$ of a triangular mesh $\mathcal{M}$ can**not** be true.

There are functions $u \in H^1(\Omega)$ w/ $u(\underline{x}^*) = \infty$

Consider a mesh $\mathcal{M}$ with a node in $\underline{x}^*$

$$\Rightarrow \quad \underbrace{I_1 u(\underline{x}^*)}_{\mathcal{M}\text{- p.w. linear}} = \infty$$

$$\Rightarrow^* \quad \| I_1 u(\underline{x}^*) \|_{L^2(\Omega)} = \infty$$

$$\Rightarrow \quad \underbrace{\| u - I_1 u \|_{L^2(\Omega)}}_{= \infty} \geq \underbrace{\| I_1 u \|_{L^2}}_{= \infty} - \underbrace{\| u \|_{L^2}}_{< \infty}$$

$\Leftarrow$

* because a finite-element function cannot blow up "very locally" like $\underline{x} \to \log|\log\|\underline{x} - \underline{x}^*\||$

①

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

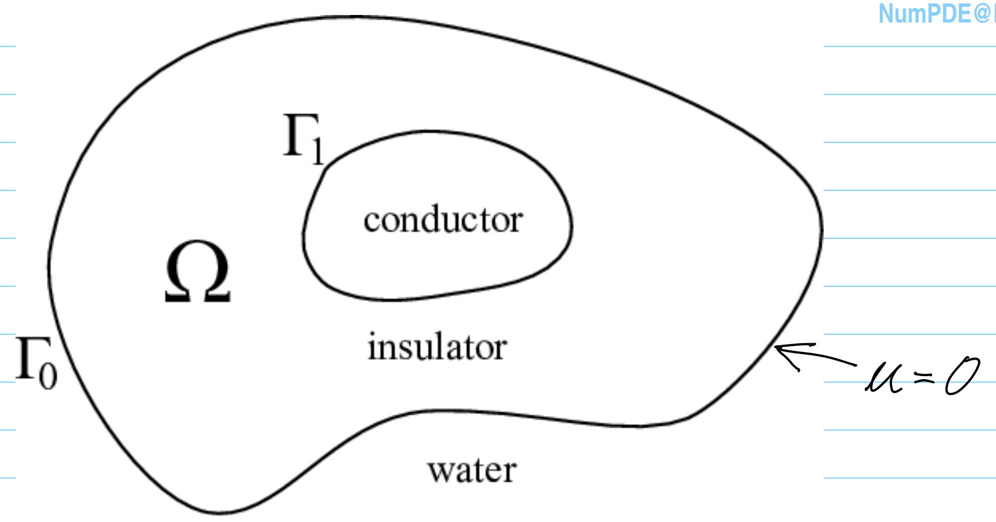
Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session

April 19, 2021

HW 1-6 e)



$$u \in V_0 : \int_\Omega \kappa(\mathbf{x})\, \mathbf{grad}\, u(\mathbf{x}) \cdot \mathbf{grad}\, v(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Gamma_1} v(\mathbf{x})\,\mathrm{d}S, \quad \forall v \in V_0, \tag{1.6.1}$$

$$V_0 := \left\{ v \in H^1(\Omega) : v|_{\Gamma_0} = 0,\ v|_{\Gamma_1} = \text{const} \right\}. \tag{1.6.2}$$

Question: Implied natural b.c. ?

Main tool: i.b.p.

Theorem 1.5.2.7. Green's first formula

For all vector fields $\mathbf{j} \in (C^1_{pw}(\overline{\Omega}))^d$ *and functions* $v \in C^1_{pw}(\overline{\Omega})$ *holds*

$$\int_\Omega \mathbf{j} \cdot \mathbf{grad}\, v\,\mathrm{d}\mathbf{x} = -\int_\Omega \operatorname{div}\mathbf{j}\, v\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{j}\cdot\mathbf{n}\, v\,\mathrm{d}S. \tag{1.5.2.8}$$

Step I: Test with $v \in C_0^\infty(\Omega)$ & i.b.p

$$\int_\Omega -\operatorname{div}(\kappa(x)\,\mathbf{grad}\,u)\cdot v\,dx = 0$$

$\Rightarrow$ PDE: $-\operatorname{div}(\kappa(x)\,\mathbf{grad}\,u) = 0$ in Ω

② Step II: Test with $v \in V_0$ & i.b.p.

$$\int_\Omega \underbrace{-\operatorname{div}(\kappa(x)\operatorname{grad} u)}_{=0 \text{ by Step I}}\, v\, dx + \int_{\cancel{\partial\Omega}\to\Gamma_1} \kappa(x)\operatorname{grad} u\cdot n\, v\, dS = \int_{\Gamma_1} v\, dS$$

$v_{|\Gamma_0} = 0$

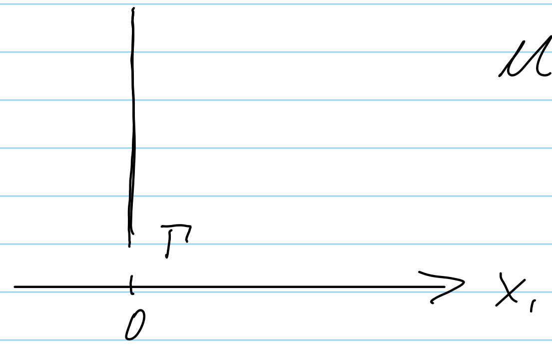
$$\Rightarrow \int_{\Gamma_1} \kappa(x)\operatorname{grad} u \cdot v\, dS = \int_{\Gamma_1} v\, dS$$

$v \in V_0 \Rightarrow$ a constant $(\to \text{set} = 1)$

$$\Rightarrow \int_{\Gamma_1} \kappa(x)\operatorname{grad} u\cdot n\, dS = |\Gamma_1|$$

A non-local total-flux boundary condition

Note: $u_{|\Gamma_1} = \text{const} \not\Rightarrow \operatorname{grad} u_{|\Gamma_1} = 0$



$u(x) = x_1 \Rightarrow \operatorname{grad} u(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Related: 1.5.3

Existence and Uniqueness of solutions for QMP

by Michael Vollenweider - Sunday, 18 April 2021, 10:58 AM

I got confused regarding the criteria for existence and uniqueness of quadratic minimization problems.

Why does the uniqueness condition (Theorem 1.2.3.31) not imply the existence (Corollary 1.2.3.26)?

Could please give a structured overview?

Theorem 1.2.3.31. Uniqueness of solutions of quadratic minimization problems

If the bilinear form $\mathsf{a} : V_0 \times V_0 \mapsto \mathbb{R}$ *is positive definite* (→ *Def. 1.2.3.27), then any solution of*

$$\underbrace{u_* = \operatorname*{argmin}_{u\in V_0} J(u) \quad , \quad J(u) := \tfrac{1}{2}\mathsf{a}(u,u) - \ell(u) + c}_{QMP} \,,$$

is unique for any linear form $\ell : V_0 \mapsto \mathbb{R}$.

Why no existence?

(i) [→ § 1.3.1.1]: $V_0 = \mathbb{Q}$, $a(u,v) = u\cdot v$, $\ell(v) = \sqrt{2}\, v$

$J(v) = \frac{1}{2}v^2 - \sqrt{2}\, v \longrightarrow$ no minimizer in $\mathbb{Q}$

(ii) Ex 1.3.0.1

We consider the quadratic functional

$$J(u) := \int_0^1 \tfrac{1}{2}u^2(x) - u(x)\,\mathrm{d}x = \tfrac{1}{2}\int_0^1 \{(u(x)-1)^2 - 1\}\,\mathrm{d}x\,,$$

▷ No minimizer in $C_0^0([0,1])$

③ **HW 3-5c)**

$a(u,v) := \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, dx + \int_{\partial\Omega} uv \, dS$, $u,v \in H^1(\Omega)$

$\hookrightarrow$ s.p.d. $\Rightarrow$ energy norm $\|\cdot\|_a$

To show: $\ell(v) := \int_\Omega f v \, dx$, $f \in L^2(\Omega)$ is bounded w.r.t. $\|\cdot\|_a$

$(*) \Leftrightarrow \exists C > 0 : |\ell(v)| \leq C \|v\|_a \quad \forall v \in H^1(\Omega)$

- C.S.I: $|\ell(v)| \leq \|f\|_{L^2} \|v\|_{L^2} \quad \forall v \in H^1(\Omega)$

$\|v\|_{L^2} \leq C \|v\|_a \Rightarrow (*)$

$\hookrightarrow$ by combining all estimates.

- Tool:

Theorem 1.8.0.20. Second Poincaré-Friedrichs inequality

If $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is bounded and connected, then

$$\exists C = C(\Omega) > 0: \quad \|u\|_0 \leq C \operatorname{diam}(\Omega) \|\mathbf{grad}\, u\|_0 \quad \forall u \in H^1_*(\Omega) .$$

$H^1_*(\Omega) := \{ v \in H^1(\Omega) : \int_\Omega v(x) dx = 0 \}$

Trick: split $v = v_* + \bar{v}$, $v_* \in H^1_*(\Omega)$, $\bar{v} \equiv \text{const}$

$(v_*, \bar{v})_{L^2} = 0 \underset{\text{Pythagoras}}{\Rightarrow} \|v\|_{L^2}^2 = \|v_*\|_{L^2}^2 + \|\bar{v}\|_{L^2}^2$

$\|v_*\|_{L^2} \overset{\text{Thm.}}{\leq} C \|\mathbf{grad}\, v\|_{L^2} \leq C \|v\|_a$

(Note: $\mathbf{grad}\, v = \mathbf{grad}\, v^*$)

$[\ \|v\|_a^2 = \|\mathbf{grad}\, v\|_{L^2}^2 + \|v\|_{L^2(\Gamma)}^2\ ]$

- Dealing with $\bar{v} \equiv \text{const}$

$\|\bar{v}\|_{L^2} = \sqrt{\frac{|\Omega|}{|\Gamma|}} \|\bar{v}\|_{L^2} \leq C \|\bar{v}\|_a$

need $\bar{v}$ here!

Tool:

Theorem 1.9.0.10. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \|u\|_{L^2(\partial\Omega)}^2 \leq C \|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) .$$

$$\begin{aligned} \|v^*\|_a^2 &= \|\mathbf{grad}\, v\|_{L^2}^2 + \|v^*\|_{L^2(\partial\Omega)}^2 \\ &\leq \|\mathbf{grad}\, v\|_{L^2}^2 + C \|v^*\|_{H^1} \cdot \|v^*\|_{L^2} \\ &\leq C \|\mathbf{grad}\, v\|_{L^2}^2 \leq C \|v\|_a^2 \end{aligned}$$

$\|\bar{v}\|_a \underset{\Delta\text{-inequ.}}{\leq} \|v\|_a + \|v^*\|_a \leq C \|v\|_a$

4

RQ 3.5.2.2.A

The local 2D trapezoidal rule on a triangular mesh $\mathcal{M}$ of a domain $\Omega \subset \mathbb{R}^2$

$$\int_\Omega \varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{K\in\mathcal{M}} \tfrac{1}{3}|K| \sum_{\ell=1}^{3} \varphi(\boldsymbol{a}_K^\ell) \qquad \boldsymbol{a}_K^\ell \text{ the vertices of } K\,,$$

is used to approximate the right-hand-side functional $\ell(v) := \int_\Omega f(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, $f \in C^0(\overline{\Omega})$, for a second-order elliptic boundary value problem, that is we use the perturbed functional

$$\ell_h(v) := \sum_{\boldsymbol{p}\in\mathcal{N}(\mathcal{M})} \omega_{\boldsymbol{p}} f(\boldsymbol{p})v(\boldsymbol{p})\,, \quad \omega_{\boldsymbol{p}} \in \mathbb{R}\,, \tag{3.5.2.3}$$

$$= \sum_{K:\,p\in K} \tfrac{1}{3}|K|$$

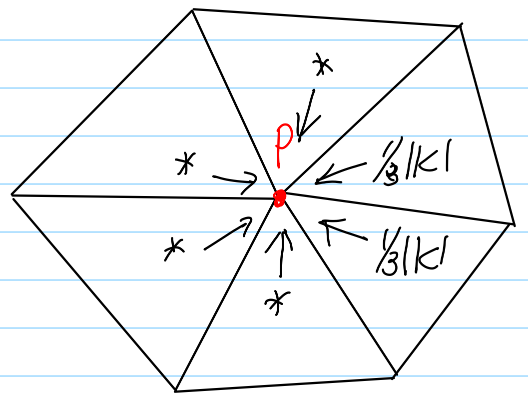
where $\mathcal{N}(\mathcal{M})$ is the set of nodes of $\mathcal{M}$.

- What are the weights $\omega_{\boldsymbol{p}}$?
- Is the functional ℓ_h from (3.5.2.3) bounded on $H^1(\Omega)$? $\longrightarrow$ No
- Is the functional ℓ_h from (3.5.2.3) continuous on $\mathcal{S}_1^0(\mathcal{M})$? $\longrightarrow$ YES because $\mathcal{S}_1^0(\mathcal{M}) \subset C^0(\overline{\Omega})$

$\to$ $\ell_h(v)$ involves point evaluation of v, which is not bounded on H^1 $\longrightarrow$ Ex. 1.2.3.45

Counterexample:

$v(x) = \log|\log\|x\||$



RQ 3.5.2.2.B

We approximate the right-hand-side functional $\ell(v) := \int_\Omega f(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, $f \in C^\infty(\overline{\Omega})$, for a linear variational problem on $H_0^1(\Omega)$ by

$$\ell_h(v) := \int_\Omega (\mathsf{I}_p f)(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \quad v \in H_0^1(\Omega)\,, \tag{3.5.2.4}$$

where $\mathsf{I}_p : C^0(\overline{\Omega}) \to \mathcal{S}_p^0(\mathcal{M})$, $p \in \mathbb{N}$, is the standard nodal interpolation operator, $\mathcal{M}$ a triangular mesh.

- Is ℓ_h from (3.5.2.4) bounded on $H_0^1(\Omega)$? $\longrightarrow$ YES: $|\ell_h(v)| \underset{\uparrow\ \text{C.S.I.}}{\leq} \underbrace{\|\mathsf{I}_p f\|_{L^2}}_{\to\ C} \|v\|_{L^2}$
- Predict the asymptotic dependence of the quantities

$$\boxed{\delta(\mathcal{M}) := \sup_{v\in H_0^1(\Omega)} \frac{|(\ell-\ell_h)(v)|}{\|v\|_{H^1(\Omega)}}}$$

on the meshwidth $h_\mathcal{M}$ on sequences of meshes obtained by uniform regular refinement.

One of the following results can help you answer the second question:

Theorem 3.3.5.6. Best approximation error estimates for Lagrangian finite elements

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded polygonal/polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices or parallelepipeds. Then, for each $k \in \mathbb{N}$, there is a constant $C > 0$ depending only on k and the shape regularity measure $\rho_\mathcal{M}$ such that

$$\inf_{v_h\in\mathcal{S}_p^0(\mathcal{M})} \|u - v_h\|_{H^1(\Omega)} \leq C\left(\frac{h_\mathcal{M}}{p}\right)^{\min\{p+1,k\}-1} \|u\|_{H^k(\Omega)} \quad \forall u \in H^k(\Omega)\,. \tag{3.3.5.7}$$

Theorem 3.5.2.5. $H^1(\Omega)$-Norm interpolation error estimates for Lagrangian finite elements

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded polygonal/polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices or parallelepipeds. Then, for each $k \in \mathbb{N}$, $k \geq 2$, $p \in \mathbb{N}$ there is a constant $C > 0$ depending only on k, the polynomial degree p, and the shape regularity measure $\rho_\mathcal{M}$ such that

$$\|u - \mathsf{I}_p u\|_{H^1(\Omega)} \leq C\,h_\mathcal{M}^{\min\{p+1,k\}-1} \|u\|_{H^k(\Omega)} \quad \forall u \in H^k(\Omega)\,,$$

where $\mathsf{I}_p : C^0(\overline{\Omega}) \to \mathcal{S}_p^0(\mathcal{M})$ is a nodal interpolation operator.

Theorem 3.5.2.6. $L^2(\Omega)$-Norm interpolation error estimates for Lagrangian finite elements

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded polygonal/polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices or parallelepipeds. Then, for each $k \in \mathbb{N}$, $k \geq 2$, $p \in \mathbb{N}$ there is a constant $C > 0$ depending only on k, the polynomial degree p, and the shape regularity measure $\rho_\mathcal{M}$ such that

$$\|u - \mathsf{I}_p u\|_{L^2(\Omega)} \leq C\,h_\mathcal{M}^{\min\{p+1,k\}} \|u\|_{H^k(\Omega)} \quad \forall u \in H^k(\Omega)\,,$$

where $\mathsf{I}_p : C^0(\overline{\Omega}) \to \mathcal{S}_p^0(\mathcal{M})$ is a nodal interpolation operator.

5

$$|(\ell-\ell_h)(v)| = \left|\int_\Omega (f - I_p f)(x)\, v(x)\, dx\right|$$

$$[\text{C.S.I.}] \leq \|f - I_p f\|_{L^2} \|v\|_{L^2}$$

$$[\text{Thm } 3.5.2.6] \text{ w/ } f \to u \leq C\, h_{\mathcal{M}}^{p+1} \|f\|_{H^p(\Omega)} \|v\|_{H^1}$$

$$\leq C' h_{\mathcal{M}}^{p+1} \|v\|_{H^1}$$

finite since $f \in C^\infty$

$$\triangleright \quad \delta(\mathcal{M}) \leq C' h_{\mathcal{M}}^{p+1}$$

RQ 3.8.0.13 A/B

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^{2,2}$, $\gamma : \Omega \to \mathbb{R}$, and $\lambda : \partial\Omega \to \mathbb{R}$ continuous, bounded, and uniformly positive coefficient functions. What is the strong (PDE) for of the boundary value problem, whose weak form reads

$$\boxed{\begin{aligned} &u \in H^1(\Omega) \\ &u = g \text{ on } \Gamma_D \end{aligned} : \int_\Omega \cancel{\boldsymbol{\alpha}(\boldsymbol{x})}\, \mathbf{grad}\, u \cdot \mathbf{grad}\, v + \cancel{\gamma(\boldsymbol{x})} u\, v\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_R} \cancel{\lambda(\boldsymbol{x})} u\, v\, \mathrm{d}S(\boldsymbol{x}) = \int_\Omega f\, v\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} h\, v\, \mathrm{d}S(\boldsymbol{x}) \quad \forall v \in H^1_{\Gamma_D}(\Omega), \quad (3.8.0.2)}$$

with $f \in L^2(\Omega)$ and the Sobolev space

$$H^1_{\Gamma_D}(\Omega) := \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \right\}, \tag{3.8.0.3}$$

based on a partition $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \Gamma_R$.

unit disk

We consider (3.8.0.2) for $\boldsymbol{\alpha} \equiv \mathbf{I}$, $\gamma \equiv 1$, $\lambda \equiv 1$, and on the unit disk domain $\Omega := \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| < 1\}$ and with $\Gamma_R := \partial\Omega$. $\Rightarrow \Gamma_N = \Gamma_D = \emptyset$

(i) Is it possible to choose the data f ~~and h~~ such that $u(\boldsymbol{x}) = \cos(\pi/2\|\boldsymbol{x}\|)$ will be the exact solution of the variational problem. If not, suggests a modification that makes it possible.

(ii) Possibly under the modification found in [(i)], determine those functions f and h that will yield that exact solution $u(\boldsymbol{x}) = \cos(\pi/2\|\boldsymbol{x}\|)$.

depends on $\|x\|$ alone

LVP: $u \in H^1(\Omega)$: $\int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v + u v\, dx + \int_{\partial\Omega} u v\, dS \overset{!}{=} \int f v\, dx \quad \forall v \in H^1(\Omega)$

(i) Go via PDE form:

$$-\Delta u + u = f \text{ in } \Omega$$

⑥ Compute f: Tool: Δ in polar coords.

$$\Delta\{x \to \varphi(\|x\|)\} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) \quad , r := \|x\|$$

$$\varphi(\xi) = \cos(\tfrac{\pi}{2}\xi)$$

$$\triangleright \quad f(x) = -\frac{1}{r}\frac{d}{dr}\left(r \cdot \sin(\tfrac{\pi}{2}r)\cdot\tfrac{\pi}{2}\right)$$

$$= -\tfrac{\pi}{2}\left(\tfrac{\pi}{2}\cos(\tfrac{\pi}{2}r) + \tfrac{1}{r}\sin(\tfrac{\pi}{2}r)\right)$$

(ii) B.C.: $\quad \mathbf{grad}\, u \cdot n + u = \cancel{0} \to h$

$\longrightarrow$ not satisfied by u !

Remedy: Introduce r.h.s in b.c.

$\Updownarrow$

extra r.h.s term: $\displaystyle\int_{\partial\Omega} h(x)v(x)\,dS(x)$ in the variational formulation.

$$\underline{\underline{//}}$$

B.C. by testing with $v \in H^1(\Omega)$ & i.b.p. & using the PDE

$\longrightarrow$ Ex 1.5.3.11 & discussion of HW 1-6e)

1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session
April 26, 2021

[Discussion of mid-term exam]

(0-1.a) (4 pts.) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\boldsymbol{a} \in \mathbb{R}^2$ fixed. For the following linear variational problem

$$u \in H_0^1(\Omega): \quad \int_\Omega (\boldsymbol{a} \cdot \mathbf{grad}\, u(\boldsymbol{x}))(\boldsymbol{a} \cdot \mathbf{grad}\, v(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} = \int_\Omega v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(\Omega) \tag{0.1.2}$$

state the related second-order elliptic boundary value problem in strong form:

$$\boxed{-\operatorname{div}(\boldsymbol{a}\boldsymbol{a}^T \operatorname{grad} u)} = \boxed{1} \quad \text{in } \Omega,$$

$$\boxed{u} = \boxed{0} \quad \text{on } \partial\Omega.$$

▲

Trick: $(\underline{a} \cdot \operatorname{grad} u)(\underline{a} \cdot \operatorname{grad} v) = (\underline{a}\underline{a}^T) \operatorname{grad} u \cdot \operatorname{grad} v$

$a(u,v) = \int_\Omega A \operatorname{grad} u \cdot \operatorname{grad} v \, dx \Rightarrow -\operatorname{div} A \operatorname{grad} u = \ldots$

$$(\underline{a}^T \operatorname{grad} u)(\underline{a}^T \cdot \operatorname{grad} v) =$$
$$= \operatorname{grad} u^T (\underline{a}\ \underline{a}^T) \operatorname{grad} v =$$
$$= ((\underline{a}\,\underline{a}^T) \operatorname{grad} v) \cdot \operatorname{grad} u =$$
$$= (aa^T) \operatorname{grad} u \cdot \operatorname{grad} v$$

②

(0-1.b) (6 pts.) We consider the linear variational problem: seek $u \in H^1(\Omega)$ such that

$$\int_\Omega (\boldsymbol{a}\cdot\boldsymbol{x})\,\mathbf{grad}\,u(\boldsymbol{x})\cdot\mathbf{grad}\,v(\boldsymbol{x}) + u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) = \int_{\partial\Omega} v(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) \quad (0.1.3)$$

for all $v \in H^1(\Omega)$, where Ω is a bounded domain and $\boldsymbol{a} \in \mathbb{R}^2$ is a given vector. State the corresponding boundary value problem:

$$\boxed{-\mathrm{div}((a\cdot x)\,\mathrm{grad}\,u) + u} = \boxed{0} \quad \text{in } \Omega,$$

$$\boxed{(a\cdot x)\,\mathrm{grad}\,u\cdot n + u} = \boxed{1} \quad \text{on } \partial\Omega.$$

B.C : Natural b.c. contained in LVP, extract them through Green's formula

$\downarrow$

impedance b.c.

(0-2.a) (4 pts.) Denote by $\mathsf{I}_\ell : C^0(\overline{\Omega}) \to \mathcal{S}_1^0(\mathcal{M}_\ell)$ the nodal interpolation operator onto the space of $\mathcal{M}_\ell$-piecewise linear Lagrangian finite-element functions. Give the maximal sets of non-negative integers from which the parameters p and q in the following asymptotic interpolation estimate can be chosen: For any

- $q \in \{\boxed{0, 1, 2}\} \subset \mathbb{N}_0$,
- $p \in \{\boxed{2, 3, 4, \ldots\ldots}\} \subset \mathbb{N}_0$,

there holds true

$$\exists C > 0: \quad \|u - \mathsf{I}_\ell u\|_{L^2(\Omega)} \le C h_\ell^q \|u\|_{H^p(\Omega)} \quad \forall u \in H^p(\Omega),\ \forall \ell \in \mathbb{N}_0. \qquad (0.2.1)$$

Corollary 3.3.3.4. Error estimate for piecewise linear interpolation in 2D

Under the assumptions/with notations of Thm. 3.3.2.21

$$\|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \le \sqrt{\tfrac{3}{8}}\, h_{\mathcal{M}}^2 |u|_{H^2(\Omega)},$$
$$|u - \mathsf{I}_1 u|_{H^1(\Omega)} \le \sqrt{\tfrac{3}{24}}\, \rho_{\mathcal{M}}\, h_{\mathcal{M}} |u|_{H^2(\Omega)}, \qquad \forall u \in H^2(\Omega).$$

Remark: $\inf_{v_h \in S_1^0(\mathcal{M})} \|u - v_h\|_{L^2(\Omega)} \le C h_\ell^q \|u\|_{H^p(\Omega)}$

In this case: $0 \le q \le \min\{1+p, 2\}$, $p \in \mathbb{N}_0$

3

In (0.2.1) $p = 0, 1$ is not possible, because I_ℓ is not defined on $L^2(\Omega) = H^0(\Omega)$ and $H^1(\Omega)$; it involves point evaluation.

$$I_\ell u = \sum_{x \in \mathcal{N}(\mathcal{M})} u(x)\, b_h^x$$

tent function

Can be $= \infty$ for some $u \in H^1(\Omega)$
($\rightarrow$ Needle & balloon example)

$p = 2$ possible:

Theorem 3.3.3.8. Sobolev embedding theorem

$$m > \frac{d}{2} \quad \Rightarrow \quad H^m(\Omega) \subset C^0(\overline{\Omega}) \quad \wedge \quad \exists C = C(\Omega) > 0: \quad \|u\|_\infty \leq C\|u\|_{H^m(\Omega)} \quad \forall u \in H^m(\Omega) .$$

continuous up to the boundary

(0-2.b) (4 pts.) In a numerical test for a Neumann problem

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2 \quad , \quad \mathbf{grad}\, u \cdot \boldsymbol{n} = h \quad \text{on } \partial\Omega \, ,$$

the data were chosen to yield the exact solution $u(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|^2)$. $\rightarrow$ smooth

We write $u_\ell \in \mathcal{S}_2^0(\mathcal{M}_\ell)$ for the finite-element Galerkin solution obtained on the mesh $\mathcal{M}_\ell$ by means of *quadratic* Lagrangian finite elements. Predict, how many steps of uniform regular refinement of $\mathcal{M}_\ell$ have to be conducted to reduce $|u - u_\ell|_{H^1(\Omega)}$ by a factor of 1000 in order to gain three decimal digits:

Required are $\boxed{5}$ refinement steps.

energy norm of FE error $= O(h_\ell^2)$

energy norm

1 step of refinement $\hat{=}$, $h \longrightarrow \frac{1}{2}h \Rightarrow$ err $\rightarrow \frac{1}{4}$ err

$$\left(\tfrac{1}{4}\right)^m \overset{!}{\leq} \frac{1}{1000}$$

$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024} \quad \rightarrow \quad 5$ steps

④

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. For a given uniformly bounded vector field $\mathbf{a} : \Omega \to \mathbb{R}^2$ we consider the **convection bilinear form**

$$(u,v) \mapsto \mathrm{b}(u,v) := \int_\Omega \mathbf{a}(\mathbf{x}) \cdot \mathbf{grad}\, u(\mathbf{x})\, v(\mathbf{x})\, \mathrm{d}\mathbf{x} \qquad (0.3.1)$$

for sufficiently smooth functions $u, v : \Omega \to \mathbb{R}$.

(0-3.a) (2 pts.) What are the largest possible Sobolev spaces V_0 and W_0 among $L^2(\Omega)$, $H^1(\Omega)$, and $H^2(\Omega)$ such that $\mathrm{b} : V_0 \times W_0 \to \mathbb{R}$ is *continuous*/bounded.

$$V_0 = \boxed{H^1(\Omega)} \quad \text{(space for } u\text{)},$$

$$W_0 = \boxed{L^2(\Omega)} \quad \text{(space for } v\text{)}.$$

(In each box, write one of $L^2(\Omega)$, $H^1(\Omega)$, or $H^2(\Omega)$.)

(0-3.b) (2 pts.) What are the dimensions of $V_{0,h}$ and $W_{0,h}$ in terms of the numbers

$\mathcal{S}_1^0(\mathcal{M})$ $\qquad$ $\mathcal{S}_0^{-1}(\mathcal{M})$

$\hat{=}$ p.w. constant

- N_V of nodes of $\mathcal{M}$,
- N_e of edges of $\mathcal{M}$,
- N_T of triangles of $\mathcal{M}$,
- N_Q of quadrilaterals of $\mathcal{M}$?

$$\dim V_{0,h} = \boxed{1} \cdot N_V + \boxed{0} \cdot N_e + \boxed{0} \cdot N_T + \boxed{0} \cdot N_Q\,,$$

$$\dim W_{0,h} = \boxed{0} \cdot N_V + \boxed{0} \cdot N_e + \boxed{1} \cdot N_T + \boxed{1} \cdot N_Q\,.$$

(0-3.c) (2 pts.) In a LEHRFEM++-based code local-to-global index mappings for $V_{0,h}$ and $W_{0,h}$ are encoded by means of **lf::assemble::UniformFEDofHandler** objects. Complete the following lines of C++ code meant to initialize such objects (`mesh_p` is a pointer to a **lf::mesh::Mesh** object for $\mathcal{M}$):

- For $V_{0,h}$

```
lf::assemble::UniformFEDofHandler dh_V(
mesh_p, {{lf::base::RefEl::kPoint(), [1] },
  {lf::base::RefEl::kSegment(), [0] },
  {lf::base::RefEl::kTria(), [0] },
  {lf::base::RefEl::kQuad(), [0] }});
```

- For $W_{0,h}$

```
lf::assemble::UniformFEDofHandler dh_W(
mesh_p, {{lf::base::RefEl::kPoint(), [0] },
  {lf::base::RefEl::kSegment(), [0] },
  {lf::base::RefEl::kTria(), [1] },
  {lf::base::RefEl::kQuad(), [1] }});
```

▲

characteristic functions of cells serve as basis

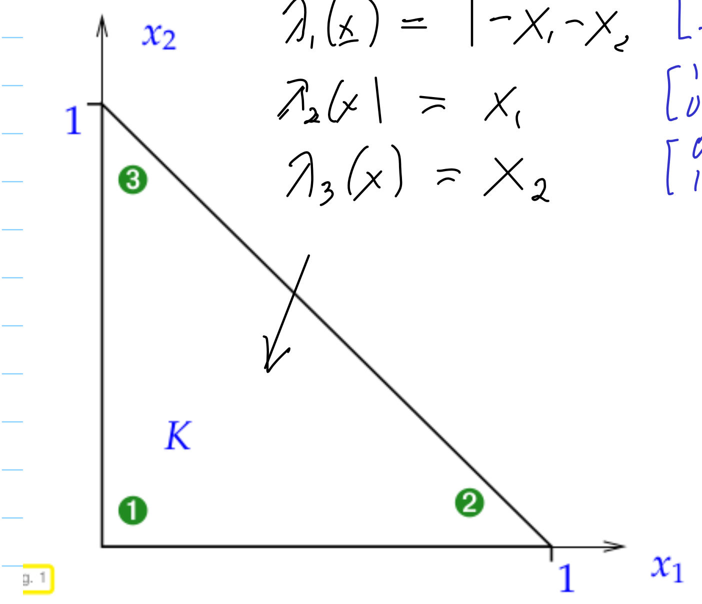
⑤

grad

$\lambda_1(x) = 1 - x_1 - x_2$ $\quad \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\lambda_2(x) = x_1$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_3(x) = x_2$ $\quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



Compute the element matrix $\mathbf{A}_K \in \mathbb{R}^{1,3}$ for the bilinear form b, with the constant vector field

$$\boldsymbol{a}(\boldsymbol{x}) \equiv \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and the finite-element Galerkin discretization introduced above on the triangle K with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, numbered as in the figure beside.

$$\mathbf{A}_K = \left[\boxed{-3/2} \quad \boxed{1/2} \quad \boxed{1} \right].$$

$$(A_K)_{1,j} = \int_K \begin{bmatrix} 1 \\ 2 \end{bmatrix} \operatorname{grad} \lambda_j \cdot 1 \, dx$$

$|K| = 1/2$

characteristic function of K

(0-3.e) (7 pts.) We implement an **ENTITY_MATRIX_PROVIDER** class for the finite-element Galerkin discretization of b as specified above:

C++ code 0.3.2: Definition of CDBLFElemMatProvider

```
template <typename MeshFunction>
class CDBLFElemMatProvider {
  public:
  // No default constructor
  CDBLFElemMatProvider() = delete;
  // Constructor takes a vectorfield argument
  explicit CDBLFElemMatProvider(MeshFunction av) : av_(std::move(av)) {}
  // The crucial interface methods for ENTITY_MATRIX_PROVIDER
  virtual bool isActive(const lf::mesh::Entity& /*cell*/) { return true; }
  Eigen::Matrix<double, 1, 3> Eval(const lf::mesh::Entity& tria);

  private:
  MeshFunction av_;
};
```

The data member av_ will contain the coefficient $\boldsymbol{x} \mapsto \boldsymbol{a}(\boldsymbol{x})$ as a **lf::mesh::utils::MeshFunctionGlobal** object, which is supposed to have an evaluation operator

```
std::vector<double>
  operator() (const lf::mesh::Entity &K,
              const Eigen::MatrixXd &refcoords);
```

Supply the missing parts of the following implementation of the crucial `Eval()` member function. It relies on the auxiliary function

```
std::pair<Eigen::Matrix<double, 2, 3>, double>
getGradBaryCoords(const lf::mesh::Entity &tria);
```

which returns, for a triangular cell passed in `tria`,

(i) a 2×3-matrix, whose columns contain the constant gradients of the barycentric coordinate functions,

(ii) the area of the triangle.

C++ code 0.3.3: Implementation of `Eval()` member function for CDBLFElemMatProvider

```
template <typename MeshFunction>
Eigen::Matrix<double, 1, 3> CDBLFElemMatProvider<MeshFunction>::Eval(
const lf::mesh::Entity& tria) {
  // Throw error in case no triangular cell
  LF_VERIFY_MSG(tria.RefEl() == lf::base::RefEl::kTria(),
  "Unsupported cell type " << tria.RefEl());
  // Fetch constant gradients of barycentric coordinate functions
  // and the area of the triangle
  auto [grad_bary_coords, area] = getGradBaryCoords(tria);
  // Reference coordinates of quadrature nodes
  const Eigen::MatrixXd refqrnodes{
    (Eigen::MatrixXd(2, 3) << 0.5, 0.5, 0.0, 0.0, 0.5, 0.5).finished()};   ∈ R^{3,3}
  std::vector<Eigen::Vector2d> av_values{av_(tria, refqrnodes)};
  Eigen::Matrix<double, 1, 3> elmat{Eigen::Matrix<double, 1, 3>::Zero()};
  for (int i = 0; i < 3; ++i) {*   -> loop over LSF
    for (int j = 0; j < 3; ++j) {
      elmat(0, i) += (grad_bary_coords.col(i)).transpose()*av_values[j];
    }
  }
  return (area/3) * elmat;
```

index into element matrix

* Loop over quadrature points

7

①

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021
(C) Seminar für Angewandte Mathematik, ETH Zürich

Q & A Session
May 3, 2021

RQ 6.3.2.33 B

Let $t \in I \mapsto \mathbf{y}(t)$, $I \subset \mathbb{R}$ an interval containing 0, denote the solution of the autonomous IVP

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) \quad , \quad \mathbf{y}(0) = \mathbf{y}_0 \; .$$

Assume that $\mathbf{f}$ is continuously differentiable.
Use the chain rule to express $\dot{\mathbf{y}}(t^*)$ and $\ddot{\mathbf{y}}(t^*)$ by means of $\mathbf{f}$ and its Jacobian.

$$\dot{y}(t^*) = f(y(t^*)) \quad \text{by ODE}$$

$$\ddot{y}(t) = \frac{d}{dt}\dot{y}(t) = \frac{d}{dt} f(y(t)) \underset{\text{chain rule}}{=} Df(y(t))\,\dot{y}(t) = \underbrace{Df(y(t))}_{\in\mathbb{R}^{N,N}}\underbrace{f(y(t))}_{\in\mathbb{R}^{N}}$$

RQ 6.3.2.33 C

Based on the answer to Question (**Q**6.3.2.33.B), determine the order of a single-step method for the autonomous ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, $\mathbf{f} : D \subset \mathbb{R}^N \to \mathbb{R}^N$ smooth, whose discrete evolution operator is given by

$$\Psi^h \mathbf{y} := \mathbf{y} + h\mathbf{f}(\mathbf{y}) + \tfrac{1}{2}h^2 \mathrm{D}\mathbf{f}(\mathbf{y})\mathbf{f}(\mathbf{y}) \; ,$$

defines a single-step method

where $\mathrm{D}\mathbf{f}(\mathbf{y}) \in \mathbb{R}^{N,N}$ is the Jacobian of $\mathbf{f}$ in $\mathbf{y} \in D$.

Order of algebraic convergence of single-step methods

Consider an IVP (6.1.3.2) with solution $t \mapsto \mathbf{y}(t)$ and a single step method defined by the discrete evolution Ψ (→ Def. 6.3.1.4). If the *one-step error along the solution trajectory* satisfies (Φ is the evolution map associated with the ODE, see Def. 6.1.4.3)

$$\left\| \Psi^h \mathbf{y}(t) - \Phi^h \mathbf{y}(t) \right\| \le Ch^{p+1} \quad \forall h \text{ sufficiently small, } t \in [0,T] \; , \tag{6.3.2.22}$$

for some $p \in \mathbb{N}$ and $C > 0$, then, usually,

$$\max_k \|\mathbf{y}_k - \mathbf{y}(t_k)\| \le \overline{C} h_{\mathcal{M}}^p \; ,$$

with $\overline{C} > 0$ independent of the temporal mesh $\mathcal{M}$: The (pointwise) discretization error *converges algebraically* with order/rate p.

②

exact evolution

One-step error $\| \Phi^h y - \Psi^h y \| = O(h^{p+1})$ for $h \to 0$

$\Rightarrow$ SSM is of order p

$\Phi^h y := \tilde{y}(h)$: $\dot{\tilde{y}}(t) = f(\tilde{y}(t))$, $\tilde{y}(0) = y$ (*)

Idea: Taylor expansion w.r.t. t.

$$\Phi^h y = \Phi^0 y + \frac{d\Phi^h y}{dh}(0)h + \tfrac{1}{2}\frac{d^2\Phi}{dh^2}(0)h^2 + O(h^3)$$

$$\overset{(*)}{=} y + \dot{\tilde{y}}(0)h + \tfrac{1}{2}\ddot{\tilde{y}}(0)h^2 + O(h^3)$$

$$\overset{B}{=} y + f(y)h + \tfrac{1}{2} Df(y)f(y)h^2 + O(h^3)$$

▷ $\Phi^h y - \Psi^h y = O(h^3)$ for $h \to 0$

▷ Order $p = 2$

RQ 6.1.4.9.D

Show that that the scalar autonomous initial-value problem

$$\dot{y} = \sqrt{y} \quad , \quad y(0) = 0 \, , \quad \text{State space: } D = \mathbb{R}_0^+$$

has at least two solutions in the state space $\mathbb{R}_0^+$ according to the following definition.

Definition Def. 6.1.1.2. Solution of an ordinary differential equation

A solution of the ODE $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ with continuous right hand side function $\mathbf{f}$ is a continuously differentiable *function* "of time t" $\mathbf{y} : J \subset I \to D$, defined on an open interval J, for which $\dot{\mathbf{y}}(t) = \mathbf{f}(t, \mathbf{y}(t))$ holds for all $t \in J$ ($\triangleq$ "*pointwise*").

How can this be reconciled with the assertion of the main theorem?

Theorem Thm. 6.1.3.16. Theorem of Peano & Picard-Lindelöf

If the right hand side function $\mathbf{f} : \hat{\Omega} \mapsto \mathbb{R}^N$ is locally Lipschitz continuous ($\to$ Def. 6.1.3.12) then for all initial conditions $(t_0, \mathbf{y}_0) \in \hat{\Omega}$ the IVP

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \quad , \quad \mathbf{y}(t_0) = \mathbf{y}_0 \, . \tag{6.1.3.2}$$

has a solution $\mathbf{y} \in C^1(J(t_0, \mathbf{y}_0), \mathbb{R}^N)$ with maximal (temporal) domain of definition $J(t_0, \mathbf{y}_0) \subset \mathbb{R}$.

Hint. Consider the function $y(t) = \left(\tfrac{1}{2}t\right)^2$.

ODE: $\dot{y} = \sqrt{y}$, solve by separation of variables

$$\Rightarrow \int_0^t \frac{\dot{y}(\tau)}{\sqrt{y(\tau)}} d\tau = \int_0^t \frac{d}{d\tau}\{2\sqrt{y(\tau)}\} d\tau = t + C \, , \; C \in \mathbb{R}$$

$$2\sqrt{y(t)} = t + C \Rightarrow y(t) = \tfrac{1}{4}(t+C)^2$$

$y(0) = 0 \Rightarrow C = 0$: solution $y(t) = \tfrac{1}{4}t^2$

③ Other solution: $y(t) = 0$

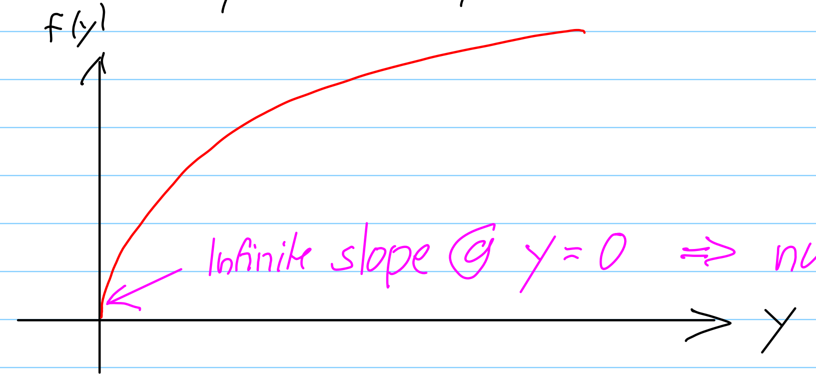
Definition 6.1.3.12. Local Lipschitz continuity $(\rightarrow)$

Let $\Omega := I \times D$, $I \subset \mathbb{R}$ an interval, $D \subset \mathbb{R}^N$, $N \in \mathbb{N}$, an open domain. A functions $\mathbf{f} : \Omega \mapsto \mathbb{R}^N$ is locally Lipschitz continuous, if for every $(t, \mathbf{y}) \in \Omega$ there is a closed box B with $(t, \mathbf{y}) \in B$ such that $\mathbf{f}$ is Lipschitz continuous on B:

$$\begin{aligned} &\forall (t,\mathbf{y}) \in \Omega: \quad \exists \delta > 0,\ L > 0: \\ &\qquad \|\mathbf{f}(\tau,\mathbf{z}) - \mathbf{f}(\tau,\mathbf{w})\| \leq L\|\mathbf{z} - \mathbf{w}\| \\ &\qquad \forall \mathbf{z}, \mathbf{w} \in D: \|\mathbf{z} - \mathbf{y}\| \leq \delta,\ \|\mathbf{w} - \mathbf{y}\| \leq \delta,\ \forall \tau \in I: |t - \tau| \leq \delta\,. \end{aligned} \tag{6.1.3.13}$$

$$|f(z) - f(w)| \leq L|z - w| \quad (N = 1)$$

Here: $f(y) = \sqrt{y}$



Infinite slope @ $y = 0$ $\Rightarrow$ not Lipschitz continuous

RQ 6.1.4.9.E

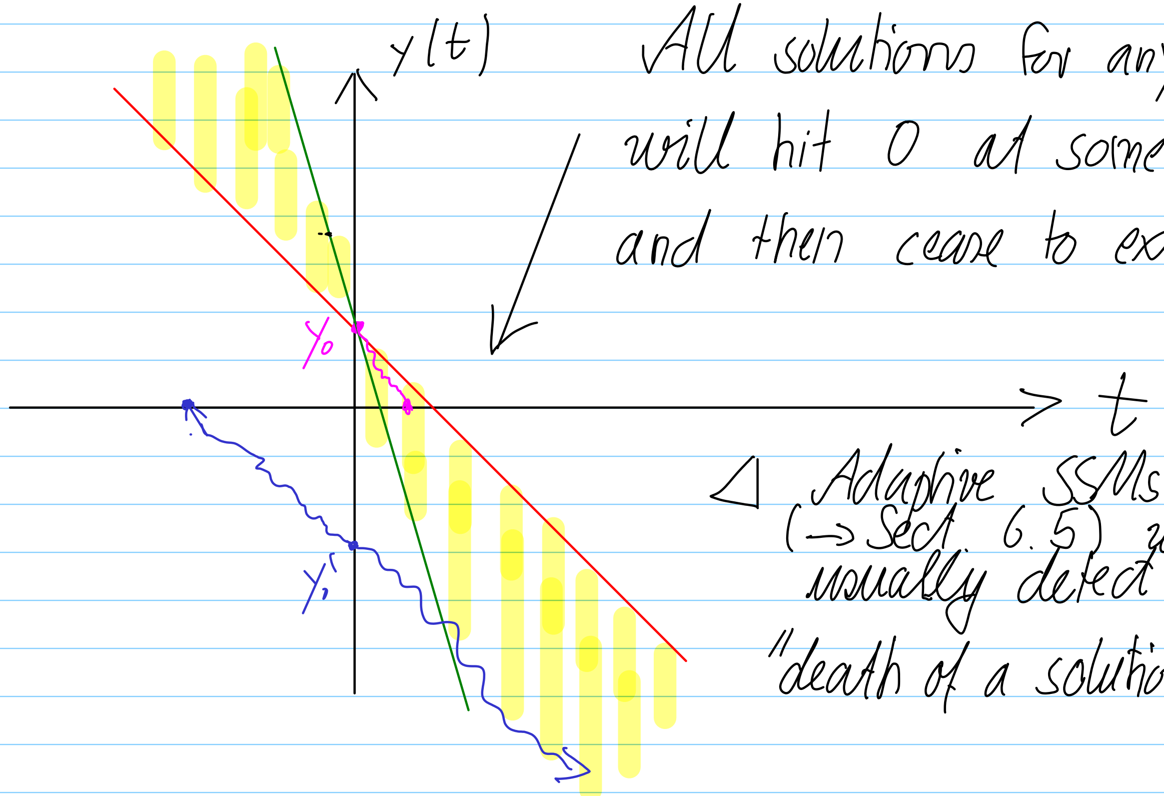
For the autonomous scalar ODE $\dot{y} = \sin\frac{1}{y} - 2$ answer the following questions

- What is the maximal state space? $\longrightarrow$ $D = \mathbb{R}\setminus\{0\}$
- Which initial values for $t_0 = 0$ will allow a global solution,
- and for which will the solution be defined for a finite time interval only?

Hint. Make use of the geometrically intuitive statement: If a differentiable function $f : [t_0, T] \to \mathbb{R}$ satisfies $\dot{f}(t) \leq C$ for all $t_0 \leq t \leq T$, then $f(t) \leq f(t_0) + Ct$.

$f(y) := \sin 1/y - 2: \quad -3 \leq f(y) \leq -1 \quad \forall y \neq 0$

$y(0) = y_0: \quad y_0 - 3t \leq y(t) \leq y_0 - t, \quad t \geq 0$



All solutions for any $y_0 \in \mathbb{R}\setminus\{0\}$ will hit 0 at some time and then cease to exist.

$\triangle$ Adaptive SSMs ($\rightarrow$ Sect. 6.5) will usually detect the "death of a solution"

$y_0 > 0$: solution exists only on a finite interval $[0, T[$.

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session
May 10, 2021

HW 7.2f) [Implicit midpoint method]

C++-code 7.2.3: Example code - solve_imp_mid ➺ GITHUB

```
template <class Function, class Jacobian>
std::vector<Eigen::VectorXd> solve_imp_mid(Function &&f, Jacobian &&Jf,
                                            double T, const Eigen::VectorXd &y0,
                                            unsigned int M) {
  std::vector<Eigen::VectorXd> res(M + 1);
  // Construct the implicit mid-point method with the class
  // implicit_RKIntegrator and execute the .solve() method.
  // Return the vector containing all steps including initial and final value.
  // Initialize implicit RK with Butcher scheme
  unsigned int s = 1;
  // Initialize coefficients for the implicit midpoint method
  Eigen::MatrixXd A(s, s);
  Eigen::VectorXd b(s);
  A << 1. / 2.;
  b << 1.;
  CrossProd::implicitRKIntegrator RK(A, b);
  res =
      RK.solve(std::forward<Function>(f), std::forward<Jacobian>(Jf), T, y0, M);
  return res;
}
```

A templated function

IMM = implicit RK-SSM* w/ Butcher scheme

s = 1 : (one stage!)

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

= 1pt Gauss-collocation SSM

② HW 7.1 b) [Stage form of expl. RK-SSM]

Definition 7.3.3.1. General Runge-Kutta single step method (*cf.* Def. 6.4.0.9)

For $b_i, a_{ij} \in \mathbb{R}$, $c_i := \sum_{j=1}^{s} a_{ij}$, $i, j = 1, \dots, s$, $s \in \mathbb{N}$, an s-stage Runge-Kutta single step method (RK-SSM) for the IVP (6.1.3.2) is defined by

$$\mathbf{k}_i := \mathbf{f}(t_0 + c_i h, \mathbf{y}_0 + h \sum_{j=1}^{s} a_{ij} \mathbf{k}_j) \,, \quad i = 1, \dots, s \quad , \quad \mathbf{y}_1 := \mathbf{y}_0 + h \sum_{i=1}^{s} b_i \mathbf{k}_i \,.$$

As before, the vectors $\mathbf{k}_i \in \mathbb{R}^N$ are called **increments**.

Equivalent stage form :

$$\mathbf{g}_i := h \sum_{j=1}^{s} a_{ij} \mathbf{k}_j \,, \quad i = 1, \dots, s \,, \quad \Leftrightarrow \quad \mathbf{k}_i = \mathbf{f}(t_0 + c_i h, \mathbf{y}_0 + \mathbf{g}_i) \,. \tag{7.3.3.7}$$

$$\mathbf{k}_i := \mathbf{f}(t_0 + c_i h, \mathbf{y}_0 + h \sum_{j=1}^{s} a_{ij} \mathbf{k}_j) \,, \quad i = 1, \dots, s \,,$$

$$\Downarrow$$

$$\mathbf{g}_i = h \sum_{j=1}^{s} a_{ij} \mathbf{f}(t_0 + c_j h, \mathbf{y}_0 + \mathbf{g}_j) \quad , \quad \mathbf{y}_1 = \mathbf{y}_0 + h \sum_{i=1}^{s} b_i \mathbf{f}(t_0 + c_j h, \mathbf{y}_0 + \mathbf{g}_i) \,. \tag{7.3.3.8}$$

Newton iteration to compute stages :

$$\mathfrak{g} = [\mathbf{g}_1, \dots, \mathbf{g}_s]^\top \in \mathbb{R}^{s \cdot N} \,,$$

$$\mathbf{g}_i := h \sum_{j=1}^{s} a_{ij} \mathbf{f}(t_0 + c_j h, \mathbf{y}_0 + \mathbf{g}_j)$$

$$\blacktriangleright \quad F(\mathfrak{g}) = \mathfrak{g} - h(\mathfrak{A} \otimes \mathbf{I}_N) \begin{bmatrix} \mathbf{f}(t_0 + c_1 h, \mathbf{y}_0 + \mathbf{g}_1) \\ \vdots \\ \mathbf{f}(t_0 + c_s h, \mathbf{y}_0 + \mathbf{g}_s) \end{bmatrix} \stackrel{!}{=} \mathbf{0} \,,$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix and $\otimes$ designates the Kronecker product introduced in .

$$\vec{g}^{(k+1)} = \vec{g}^{(k)} - DF(\vec{g}^{(k)})^{-1} F(\vec{g}^{(k)})$$

$$A \otimes I_N = \begin{bmatrix} a_{11} I_N & \cdots & a_{1s} I_N \\ \vdots & & \\ a_{s1} I_N & \cdots & a_{ss} I_N \end{bmatrix} \in \mathbb{R}^{Ns, Ns}$$

$$F\left(\begin{bmatrix} g_1 \\ \vdots \\ g_s \end{bmatrix}\right) = \begin{bmatrix} F_1(\vec{g}) \\ \vdots \\ F_s(\vec{g}) \end{bmatrix} = \left[g_i - h \sum_{j=1}^{s} a_{ij} f(t_0 + c_j h, y_0 + g_j) \right]_{i=1}^{s}$$

$$DF(\vec{g}) = \left[\frac{\partial F_i}{\partial g_k}(g_1, \dots, g_s) \right]_{i,k=1}^{s}$$

$$= \left[\delta_{ik} I_N - h a_{ik} \frac{\partial f}{\partial y}(t_0 + c_k h, y_0 + g_k) \right]_{i,k=1}^{s}$$

$$= \begin{bmatrix} I_N - h a_{11} \frac{\partial f}{\partial y}(t_0 + c_1 h, y_0 + g_1) & -h a_{12} \frac{\partial f}{\partial y}(t_0 + c_2 h, y_0 + g_2) & \cdots \\ -h a_{21} \frac{\partial f}{\partial y}(t_0 + c_1 h, y_0 + g_1) & & \\ \vdots & & \end{bmatrix}$$

③ RQ 6.3.1.16 B

There is a connection between numerical integration (the design and analysis of numerical methods for the solution of initial-value problems for ODEs) and numerical quadrature (study of numerical methods for the evaluation of integrals).

- Explain, how a class of single-step methods for the solution of scalar initial-value problems

$$\dot{y} = f(t,y) \quad , \quad y(t_0) = y_0 \in \mathbb{R} \,,$$

can be used for the approximate evaluation of integrals $\int_a^b \varphi(\tau)\,\mathrm{d}\tau$, $\varphi : [a,b] \to \mathbb{R}$.
- If the considered single-step methods are of order p, what does this mean for the induced quadrature method.
- Which quadrature formula does the implicit midpoint method yield?

By fundamental theorem of calculus:

$\dot{y} = f(t,y) \Rightarrow y(b) = y(a) + \int_a^b f(t,y(t))\,dt$

If $f = f(t)$, $y(a) = 0 \Rightarrow y(b) = \int_a^b f(t)\,dt$

▷ Apply SSM to $\dot{y} = f(t)$, $y(a) = 0$

$y_M \approx \int_a^b f(t)\,dt$ after M steps

SSM of order $p \Rightarrow y_M - y(b) = O(h^p)$

for equidistant timesteps

▷ SSM provides a composite quadrature rule that, on equidistant mesh, has alg. cvg. of order p.

Remark: Recall order p of a quadrature rule

$\leftrightarrow$ degree $p-1$ of polynomial exactness

[RK-SSM: order of SSM $\Rightarrow$ order of QR]

IMM: $y_1 = y_0 + f(\tfrac{1}{2}(t_0+t_1), \tfrac{1}{2}(y_0+y_1))$

$\dot{y} = f(t)$: $y_1 = y_0 + h f(\tfrac{1}{2}(t_0+t_1))$

$$\int_a^b f(t)\,dt \approx \sum_{j=1}^{M} (t_j - t_{j-1})\, f(\tfrac{1}{2}(t_j + t_{j-1}))$$

$\hat{=}$ Composite midpoint quadrature rule

4 RQ 6.3.1.16 D

We have seen three simple single-step methods for the autonomous ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, $\mathbf{f} : D \subset \mathbb{R}^N \to \mathbb{R}^N$, here defined by describing a the first step $\mathbf{y}_0 \to \mathbf{y}_1$ with stepsize $h \in \mathbb{R}$ ("sufficiently small"):

- The explicit Euler method:

$$\mathbf{y}_1 = \mathbf{y}_0 + h\mathbf{f}(\mathbf{y}_0) .$$

- The implicit Euler method:

$$\mathbf{y}_1: \quad \mathbf{y}_1 = \mathbf{y}_0 = h\mathbf{f}(\mathbf{y}_1) .$$

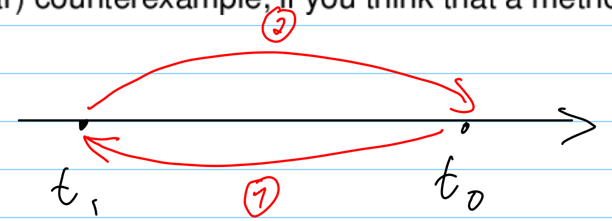
- The implicit midpoint method:

$$\mathbf{y}_1: \quad \mathbf{y}_1 = \mathbf{y}_0 + h\mathbf{f}(\tfrac{1}{2}(\mathbf{y}_0 + \mathbf{y}_1)) .$$

For which methods does the associated discrete evolution operator $\mathbf{\Psi} : [-\delta, \delta] \times D \to D$, $\delta > 0$ sufficiently small, satisfy

$$\mathbf{\Psi}^h \circ \mathbf{\Psi}^{-h} = \mathsf{Id} \quad \forall h \in [0, \delta] \ ? \tag{6.3.1.17}$$

Try to find a simple (scalar) counterexample, if you think that a method does not have property (6.3.1.17).



(6.3.1.17) satisfied by every exact evolution operator

EEUL: Study for $\dot{y} = y$, $y(0) = 1$

$y_1 = 1 - h$, $y_2 = 1 - h + h(1-h) = 1 - h^2 \neq 1$

Not satisfied!

IEUL: Study for $\dot{y} = y$, $y(0) = y_0$

$y_1 = y_0 - h y_1 \Rightarrow y_1 = \frac{1}{1+h} y_0$

$y_2 = \frac{1}{1+h} y_0 + h y_2 \Rightarrow y_2 = \frac{1}{1-h} \frac{1}{1+h} y_0$

$= \frac{1}{1-h^2} y_0 \neq y_0$

Not satisfied!

IMM: Study for $\dot{y} = y$, $y(0) = 1$

$y_1 = 1 - h \tfrac{1}{2}(1 + y_1) \Rightarrow y_1 = \frac{1}{1+\frac{1}{2}h} \cdot (1 - \tfrac{1}{2}h)$

$y_2 = \frac{1-\frac{1}{2}h}{1+\frac{1}{2}h} + h \tfrac{1}{2} \left(\frac{1-\frac{1}{2}h}{1+\frac{1}{2}h} + y_2 \right)$

$\Rightarrow y_2 = \frac{1}{1-\frac{1}{2}h} \left(\frac{1-\frac{1}{2}h}{1+\frac{1}{2}h} + \tfrac{1}{2}h \frac{1-\frac{1}{2}h}{1+\frac{1}{2}h} \right) = 1$

IMM: $y_1 = y_0 - hf(\tfrac{1}{2}(y_0 + y_1))$

$\Rightarrow y_0 = y_1 + hf(\tfrac{1}{2}(y_1 + y_0))$

$y_2 = y_1 + hf(\tfrac{1}{2}(y_1 + y_2))$

$\Rightarrow y_0 = y_2$ if h sufficiently small

⑤ **RQ 6.4.0.18 B** [Autonomization]

Recall that by "autonomization" the initial value problem

$$\dot{\mathbf{y}} = \mathbf{f}(t,\mathbf{y}) \quad , \quad \mathbf{y}(t_0) = \mathbf{y}_0 \,, \tag{6.4.0.19}$$

with $\mathbf{f} : I \times D \to \mathbb{R}^N$ can be converted into the equivalent IVP for the extended state $\mathbf{z} = [z_1, \dots, z_N, z_{N+1}]^\top := [\mathbf{y}\ t]^\top \in \mathbb{R}^{N+1}$:

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}) \ , \quad \mathbf{g}(\mathbf{z}) := \begin{bmatrix} \mathbf{f}(z_1,\dots,z_N) \\ 1 \end{bmatrix} \quad , \quad \mathbf{z}(0) = \begin{bmatrix} \mathbf{y}_0 \\ t_0 \end{bmatrix} . \tag{6.4.0.20}$$

Let us apply the same 2-stage explicit Runge-Kutta method to (6.4.0.19) and (6.4.0.20). When will both approaches produce the same sequence of states $\mathbf{y}_k \in D$?

2-stage explicit RK-SSM for $\dot{y} = f(t,y)$:

$$\begin{array}{c|cc} c_1 & 0 & 0 \\ c_2 & a_{21} & 0 \\ \hline & b_1 & b_2 \end{array}$$

$$k_1 = f(t_0 + c_1 h, y_0)$$

$$k_2 = f(t_0 + c_2 h, y_0 + h a_{21} k_1)$$

$$y_1 = y_0 + h(b_1 k_1 + b_2 k_2)$$

$$= y_0 + h b_1 f(t_0, y_0) + h b_2 f(t_0 + c_2 h, y_0 + h a_{21} f(t_0, y_0))$$

Same RK-SSM for $\dot{z} = g(z)$

$$\tilde{k}_1 = g(z_0) = \begin{bmatrix} f(t_0, y_0) \\ 1 \end{bmatrix} \qquad z_0 = \begin{bmatrix} y_0 \\ t_0 \end{bmatrix}$$

$$\tilde{k}_2 = g(z_0 + h a_{21} \tilde{k}_1) = \begin{bmatrix} f(t_0 + h a_{21}, y_0 + h a_{21} f(t_0, y_0)) \\ 1 \end{bmatrix}$$

$$z_1 = z_0 + h(b_1 \tilde{k}_1 + b_2 \tilde{k}_2)$$

$$= \begin{bmatrix} y_0 \\ t_0 \end{bmatrix} + h\left(b_1 \begin{bmatrix} f(t_0, y_0) \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} f(t_0 + h a_{21}, y_0 + h a_{21} f(t_0, y_0)) \\ 1 \end{bmatrix}\right)$$

(blue) = (green), if $t_1 = t_0 + h(b_1 + b_2) \Leftrightarrow b_1 + b_2 = 1$

$$c_2 = a_{21}$$

Consistency condition for RK-SSMs

□

1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session

May 17, 2021

HW 9-3f)

Abstract parabolic evolution problem :

$u : [0,T] \to V_0 \; : \; m(\dot{u}, v) + a(u,v) = 0 \quad \forall v \in V_0$

$u(0) = u_0 \in V_0$

s.p.d. bilinear forms

Poincaré-Friedrichs-type inequality (PFI)

$\exists \gamma > 0 \; : \; a(u,u) \geq \gamma \, m(u,u) \quad \forall u \in V_0$

Lemma 9.2.3.8. Decay of solutions of parabolic evolutions

For $f \equiv 0$ the solution $u(t)$ of (9.2.2.4) satisfies

$$\|u(t)\|_m \leq e^{-\gamma t}\|u_0\|_m \quad , \quad \|u(t)\|_a \leq e^{-\gamma t}\|u_0\|_a \quad \forall t \in]0,T[\, ,$$

where $\gamma > 0$ is the constant from (9.2.3.6), and $\|\cdot\|_a$, $\|\cdot\|_m$ stand for the energy norms induced by $\mathsf{a}(\cdot,\cdot)$ and $\mathsf{m}(\cdot,\cdot)$, respectively.

MOL-ODE based on spatial Galerkin discretization, V_0 replaced with $V_{0,h} \subset V_0$, $V_{0,h} = \mathrm{Span}\{b_h^1, \dots, b_h^N\}$

$$M \dot{\vec{\mu}}(t) + A \vec{\mu}(t) = 0$$

Implicit Euler timestepping : $\mu(t) \in \mathbb{R}^N$

$$M(\vec{\mu}^{(k)} - \vec{\mu}^{(k-1)}) = -\tau A \vec{\mu}^{(k)}$$

②

M, A s.p.d. $\Rightarrow$ norms $\|\cdot\|_M, \|\cdot\|_A$

$\vec{\mu}^{(k)T}\cdot | \qquad M(\vec{\mu}^{(k)} - \vec{\mu}^{(k-1)}) = -\tau A \vec{\mu}^{(k)}$

$\Rightarrow \quad \vec{\mu}^{(k)T} M \vec{\mu}^{(k)} - \vec{\mu}^{(k)T} M \vec{\mu}^{(k-1)} = -\tau\, \vec{\mu}^{(k)T} A \vec{\mu}^{(k)}$

$\Leftrightarrow \quad \|\vec{\mu}^{(k)}\|_M^2 = \vec{\mu}^{(k)T} M \vec{\mu}^{(k-1)} - \tau\, \vec{\mu}^{(k)T} A \vec{\mu}^{(k)}$

PFI $\Rightarrow \quad \vec{v}^T A \vec{v} \geq \gamma\, \vec{v}^T M \vec{v} \quad \forall \vec{v} \in \mathbb{R}^N$

$\|\vec{\mu}^{(k)}\|_M^2 \leq \vec{\mu}^{(k)T} M \vec{\mu}^{(k-1)} - \tau\gamma\, \vec{\mu}^{(k)T} M \vec{\mu}^{(k)}$

CSI $\Rightarrow \quad |\vec{v}^T M \vec{\mu}| \leq \|\vec{v}\|_M \cdot \|\vec{\mu}\|_M$

$\|\vec{\mu}^{(k)}\|_M^{\cancel{2}} \leq \cancel{\|\vec{\mu}^{(k)}\|_M} \|\vec{\mu}^{(k-1)}\|_M - \tau\gamma \|\vec{\mu}^{(k)}\|_M^{\cancel{2}}$

$\|\mu^{(k)}\|_M \leq \dfrac{1}{1+\tau\gamma} \|\vec{\mu}^{(k-1)}\|_M$

Exponential decay!

HW 9-2d)

HW 9.2 d)

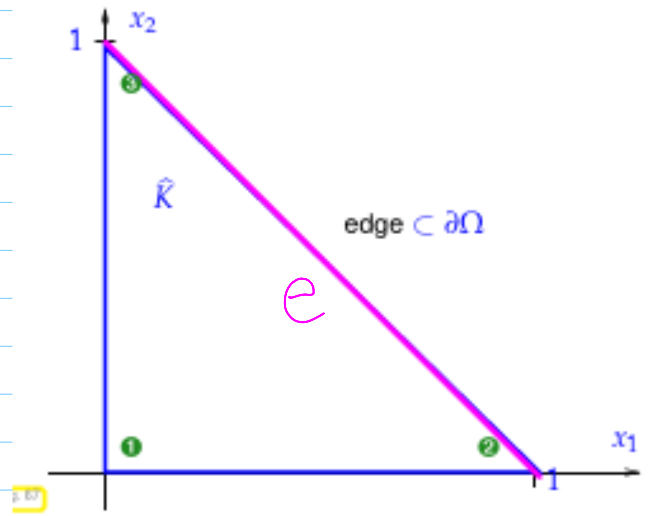
by Paolo Bottoni - Monday, 17 May 2021, 10:08 AM

Hello there

I had a problem with this exercise:

how do we arrive to the pink part? is this important to know?

(9-2.d) (20 min.) [depends on Sub-problem (9-2.a)]



Compute the $\mathcal{S}_1^0(\mathcal{M})$ element (stiffness) matrix $\mathbf{A}_{\hat{K}}$ corresponding to $a(\cdot,\cdot)$ for the unit triangle $\hat{K}$ with vertices $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. Assume that the edge connecting $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$ forms part of $\partial\Omega$ and that the coefficient c is constant along this edge.

typo!

$a(u,v) := \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \,dx + \int_{\partial\Omega} c(x)\, u v \,dS$

$u, v \in H^1(\Omega)$

this part of $a(\cdot,\cdot)$ will be relevant for computing $a_K(b_K^2, b_K^3)$, $a_K(b_K^2, b_K^2)$, $a_K(b_K^3, b_K^3)$

③ 3×3 ~ Element matrix:

$$A_K = \begin{bmatrix} & * & * \\ & * & * \end{bmatrix}$$

$$\mathbf{A}_{\widehat{K}} = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{\sqrt{2}}{6}c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

$$= c\int_e \lambda_K^i \lambda_K^j \, dS$$

See Ex. 2.7.4.37 & (2.7.4.40)

$\lambda_K^2(\underline{x}) = x_1$, $\lambda_K^3(\underline{x}) = x_2$

Parametrization of e: $t \in [0,1] \longrightarrow \begin{bmatrix} t \\ 1-t \end{bmatrix} =: \gamma(t)$

$$\int_e \lambda_K^2 \lambda_K^3 \, dS = \int_0^1 t\,(1-t)\,\left\|\frac{d\gamma}{dt}\right\| \, dt$$

$$= 1/6 \cdot \sqrt{2}$$

RQ 7.4.0.10.A

The implicit midpoint single-step method applied to the autonomous ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ and with timestep h leads to the recursion

$$\mathbf{y}_{k+1}: \quad \mathbf{y}_{k+1} = \mathbf{y}_k + h\mathbf{f}(\tfrac{1}{2}(\mathbf{y}_k + \mathbf{y}_{k+1}))\,.$$

Derive the defining equation of the semi-implicit variant, which arises from solving the defining equation for $\mathbf{y}_{k+1}$ by a single Newton step with initial guess $\mathbf{y}_k$.

$\underline{y}_{k+1}$ solves $F(\underline{x}) = \underline{0}$ with

$$F(\underline{x}) := \underline{x} - \underline{y}_k - hf(\tfrac{1}{2}(\underline{x} + \underline{y}_k))$$

$$DF(\underline{x}) = \underline{I} - h\,Df(\tfrac{1}{2}(\underline{x} + \underline{y}_k)) \cdot \tfrac{1}{2}$$

$$\underline{x}^{(1)} = \underline{x}^{(0)} - DF(\underline{x}^{(0)})^{-1} F(\underline{x}^{(0)})$$

Initial guess: $\underline{x}^{(0)} = \underline{y}_k$

$$\underline{y}_{k+1} = \underline{x}^{(1)} = \underline{y}_k + (I - h\tfrac{1}{2}Df(\underline{y}_k))^{-1} hf(\underline{y}_k)$$

4

RQ 7.4.0.10. B

A Rosenbrock-Wanner (ROW) single-step method for the autonomous ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ can be defined by

$$(\mathbf{I} - ha_{ii}\mathbf{J})\mathbf{k}_i = \mathbf{f}(\mathbf{y}_0 + h\sum_{j=1}^{i-1}(a_{ij}+d_{ij})\mathbf{k}_j) - h\mathbf{J}\sum_{j=1}^{i-1} d_{ij}\mathbf{k}_j\,, \quad \mathbf{J} = D\mathbf{f}(\mathbf{y}_0)\,, \tag{7.4.0.9}$$

$$\mathbf{y}_1 := \mathbf{y}_0 + h\sum_{j=1}^{s} b_j\mathbf{k}_j\,.$$

Derive its stability functions for $s = 2$.

Apply ROW-method to $\dot{y} = \lambda y \Rightarrow J = \lambda$

$$(1 - ha_{11}\lambda)\,k_1 = \lambda y_0$$

$$(1 - ha_{22}\lambda)\,k_2 = \lambda(y_0 + h(a_{21}+d_{21})k_1) - h\lambda d_{21}k_1$$

$$k_1 = \frac{\lambda}{1-ha_{11}\lambda}\,y_0$$

$$k_2 = \frac{1}{1-ha_{22}\lambda}\left(\lambda y_0 + h\lambda a_2 \frac{\lambda y_0}{1-ha_{11}\lambda}\right)$$

$z := \lambda h$

$$y_1 = \underbrace{\left(1 + \frac{b}{1-za_{11}}z + \frac{b_2}{1-za_{22}}\left(z + \frac{a_2 z^2}{1-za_{22}}\right)\right)}_{= S(z)} y_0$$

RQ 9.2.2.10. B

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\rho, \kappa : \Omega \to \mathbb{R}$ uniformly positive coefficient functions. Derive the spatial variational formulation for the parabolic initial value problem

$$\frac{\partial}{\partial t}(\rho(\mathbf{x})u) - \operatorname{div}(\kappa(\mathbf{x})\,\mathbf{grad}\,u) = 0 \quad \text{in} \quad \tilde{\Omega} := \Omega\times]0,T[\,,$$

impedance type b.c. $\longrightarrow$ $$\boxed{\mathbf{grad}\,u(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) + \frac{\partial u}{\partial t}(\mathbf{x},t) = 0} \quad \text{for} \quad (\mathbf{x},t) \in \partial\Omega\times]0,T[\,,$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \Omega\,,$$

Strong form

for given initial data $u \in H^1(\Omega)$.

Theorem 1.5.2.7. Green's first formula

For all vector fields $\mathbf{j} \in (C^1_{pw}(\overline{\Omega}))^d$ and functions $v \in C^1_{pw}(\overline{\Omega})$ holds

$$\int_\Omega \mathbf{j}\cdot\mathbf{grad}\,v\,d\mathbf{x} = -\int_\Omega \operatorname{div}\mathbf{j}\,v\,d\mathbf{x} + \int_{\partial\Omega}\mathbf{j}\cdot\mathbf{n}\,v\,dS\,. \tag{1.5.2.8}$$

Approach: Multiply with $v \in V_0 = H^1(\Omega)$ & i.b.p.
[t treated as a passive parameter]

$$\int_\Omega \partial_t(\rho(x)u)\,v(x) - \operatorname{div}(\kappa(x)\,\mathbf{grad}\,u)\cdot v(x)\,dx = 0 \quad \forall v \in H^1(\Omega)$$

i.b.p. $$\frac{d}{dt}\int_\Omega \rho(x)\,u(x,t)\,v(x)\,dx + \int_\Omega \kappa(x)\,\mathbf{grad}_x u(x,t)\cdot\mathbf{grad}\,v(x)\,dx$$
$$- \int_{\partial\Omega} \kappa(x)\underbrace{\mathbf{grad}\,u(x)\cdot n(x)}_{= -\partial_t u(x,t)}\,v(x)\,dS(x) = 0 \quad \Rightarrow -\frac{d}{dt}\int_{\partial\Omega}\kappa(x)\,u\cdot v\,dS(x)$$

⑤ RQ 9.2.2.10.C

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\rho, \kappa : \Omega \times]0,T[\to \mathbb{R}$ be uniformly positive *time-dependent* coefficient functions. Derive the correct spatial variational formulation for the parabolic initial value problem

$$\frac{\partial}{\partial t}(\rho(\boldsymbol{x},t)u) - \operatorname{div}(\kappa(\boldsymbol{x},t)\,\mathbf{grad}\,u) = 0 \quad \text{in} \quad \widetilde{\Omega} := \Omega \times]0,T[\,,$$

Test functions $v \in H_0^1(\Omega)$ $\longleftarrow$

$$u(\boldsymbol{x},t) = 0 \quad \text{for} \quad (\boldsymbol{x},t) \in \partial\Omega \times]0,T[\,,$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x} \in \Omega\,,$$

for given initial data $u_0 \in H_0^1(\Omega)$.

$\triangledown_0$ No change despite time-dependent coefficients

$$\frac{d}{dt}\int_\Omega \rho(x,t)\,v(x)\,dx + \int_\Omega \kappa(x,t)\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v\,dx = 0 \quad \forall v \in H_0^1(\Omega)$$

Generalized eigenvalues :

Generalized EVP : $\quad A\vec{\mu} = \lambda M\vec{\mu} \quad (*)$

$[A, M \in \mathbb{R}^{N,N}]$ $\qquad$ M regular

$$(*) \iff \underbrace{M^{-1}A\vec{\mu} = \lambda\vec{\mu}}_{\text{std. EVP}}$$

$\ulcorner$ $\vec{\xi} := M\vec{\mu}$

$AM^{-1}\vec{\xi} = \lambda\vec{\xi}$ $\lrcorner$

$\triangleright$ Suppl. 9.2.7.20 :

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session
May 31, 2021

11.2.4 Jump Conditions

by Michael Vollenweider - Thursday, 20 May 2021, 11:41 AM

Could you please explain the pillbox thought experiment in a little more detail?

Thank you!

→ See augmented lecture document !

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

Prof. R. Hiptmair, SAM, ETH Zurich

Spring Term 2021

Q & A Session

Exam Prep, Feb 4, 2022

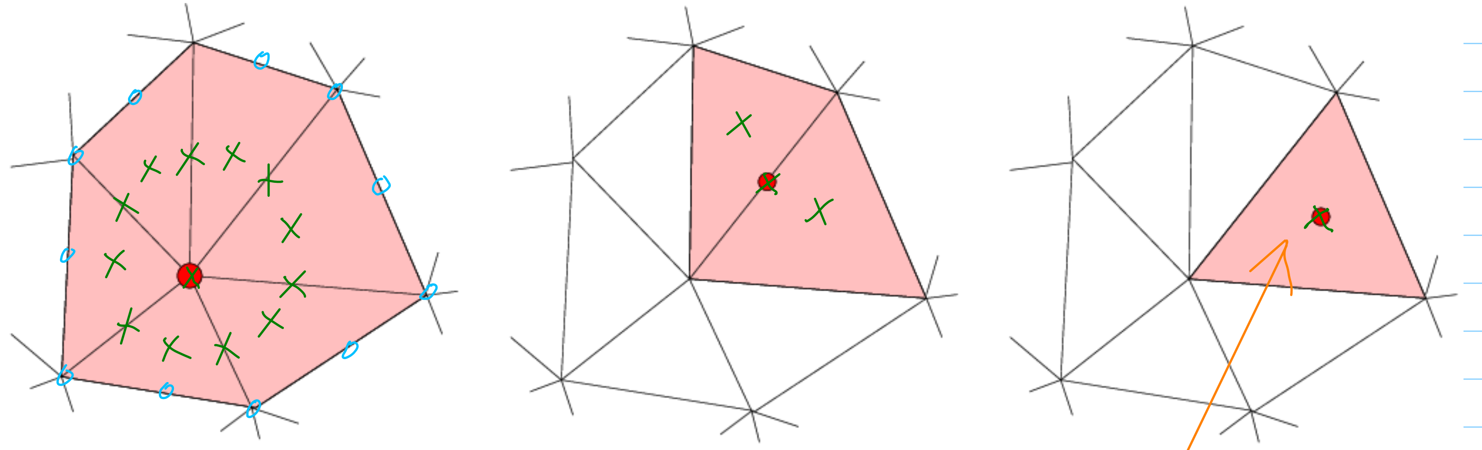
Q: Shape functions and geometric entities

→ Sect 2.5.3 & § 2.7.4.13

Hi,

could you explain again the difference between a global/local shape function **covering** an entity and a GSF **associated** with a geometric entity?
Maybe you can use the triangle in exercise 2.9 c) as an example.

X ≙ geometric entities covered by the global shape function associated with •



Support of node-associated basis function, *cf.* Fig. 99 | Support of edge-associated basis function | Support of cell-associated basis function

Note: Every global/local shape function is associated with a unique geometric entity

Example: $V_N = \{ v_N \in L^2(\Omega) : v_{N|K} \in \mathcal{P}_1(\mathbb{R}^2) \ \forall K \in \mathcal{M} \}$

→ Three global shape functions associated with every cell

Issue: Meaning of "covering" for discontinuous FE spaces

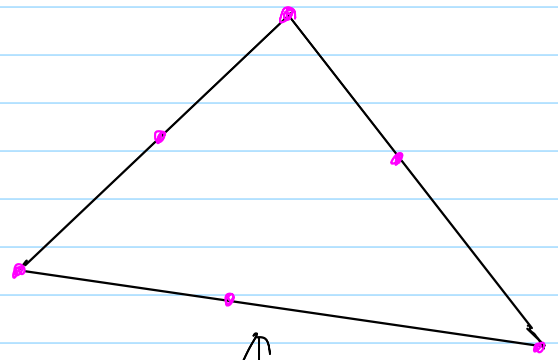
Related: LF++ DofHandler member functions

§ 2.7.4.13

{ NumLocalDofs()
GlobalDofIndices() } $\longleftrightarrow$ global shape functions covering the argument entity

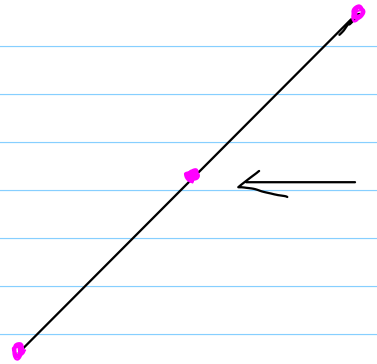
Example: $S_2^0(\mathcal{M})$: quadratic Lagrangian FE

[1 GSF associated with each vertex
1 GSF associated with each edge]



6 GSF covering every triangle

$\hat{=}$ set of all shape functions associated with subentities



$\leftarrow$ 3 GSF covering an edge

$\leftarrow$ only this GSF is associated with the edge

{ NumInteriorDofs()
InteriorGlobalDofIndices() } $\longleftrightarrow$ GSF associated with the entity passed as argument.

Notation:

Lagrangian FE spaces: $S_p^0(\mathcal{M})$

Discontinuous FE space: $S_p^{-1}(\mathcal{M})$

I also have a question. Could you explain the difference between a functor and a mesh function and why we sometimes have to use mesh functions instead of functor ?

functor $\hat{=}$ data types offering a method
X operator() (ARG) const
[also λ-fncts.]

LF++ MeshFunction $\hat{=}$ a special functor type
(used, e.g., to wrap coefficients & source functions)

5

7