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ETH Lecture 401-0674-00L Numerical Methods for (Partial) Differential Equations

Numerical Methods for (Partial) Differential Equations

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Mid-Term Exam Discussion

It is highly desirable that questions are *submitted* at least a few hours *before* the start of the Q&A session so that the lecturer has the opportunity to structure his or her answer. Submission of questions should be done through dedicated DISCUNA *chat channels*. A separate channel has been set up for each week in which a regular Q&A session will take place.

Important links and resources:

(1) Course directory page: https://www.vorlesungen.ethz.ch/Vorlesungsverzeichnis/lerneinheit.view?lerneinheitId=188112&semkez=2025S&ansicht=LEHRVERANSTALTUNGEN&lang=de
(2) Moodle page: https://moodle-app2.let.ethz.ch/course/view.php?id=24006
(3) Lecture document: https://www.sam.math.ethz.ch/~grsam/NUMPDEFL/NUMPDE.pdf
(4) Homework problem collection: https://people.math.ethz.ch/~grsam/NUMPDEFL/HOMEWORK/NPDEFL_Problems.pdf
(5) Course code repository: https://gitlab.math.ethz.ch/ralfh/NPDERepo
(6) Course repository: https://people.math.ethz.ch/~grsam/NUMPDEFL/
(7) Course polybox folder: https://polybox.ethz.ch/index.php/s/Mn25THrx1yjvcPw , PW: NPDE25 (contains course videos and tablet notes)
(8) Discuna join link: https://app.discuna.com/invite/9QLX7oxbfNl22FceUwEM

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I. Feedback "bQm Meeting"

1. More hours of tutorial classes (→ 3-4 h)
2. Move Friday tutorial class [explore]
3. Low Q&A attendance due to difficulties in keeping up with the course
   → Questions on earlier topics welcome
4. Bad: Course materials scattered over several online repositories
   → Duplicated content in PolyBox, grscam, Moodle videos
5. Chapter summaries
   → Start from tablet notes!
6. Request: Solutions of RQs
7. Request: Level of difficulty of RQs
8. Inconsistent estimated times to solution for HW problems

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II. Mid-term Problems

Problem 0-1: Strong and Weak Form of Elliptic BVPs

Elliptic boundary value problems as part of mathematical models come in various guises: some are stated as boundary value problems for PDEs in **strong form**, some are given in **weak form** as variational problems. Both forms can be converted into each other and this is the focus of this problem.

This problems is connected with [Lecture → Section 1.5] and [Lecture → Section 1.8].
▷ problem name/problem code folder: StrongWeakEllBVP

(0-1.a) (6 pts.) State the complete variational form (in appropriate Sobolev spaces) of the following second-order elliptic boundary value problem on a computational domain $\Omega \subset \mathbb{R}^2$:

$$-\left(\frac{\partial^2 u}{\partial x_1{}^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2{}^2}\right) + \|\mathbf{x}\|^2 u = 0 \quad \text{in} \quad \Omega ,$$

$$u = 1 \quad \text{on} \quad \partial\Omega . \quad [\text{ Dirichlet b.c }]$$

essential b.c.

$u \in \boxed{H^1(\Omega) : u|_{\partial\Omega} = 1}$:

$$\int_\Omega \boxed{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \operatorname{grad} u \cdot \operatorname{grad} v + \|x\|^2 u v} \, d\mathbf{x} = \boxed{0}$$

$\forall v \in \boxed{H_0^1(\Omega)}$.

If PDE $\quad -\operatorname{div} A \operatorname{grad} u + c\,u$

→ Blf. in LVP

$$a(u,v) = \int_\Omega A \operatorname{grad} u \cdot \operatorname{grad} v + c\,u v \, dx$$

$$= -\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_2}\right)$$

$$= -\operatorname{div}\Big(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \operatorname{grad} u\Big)$$

(4)

(0-1.b) (6 pts.) Determine the boundary value problem in strong from, whose variational/weak form reads:

Seek $u \in H^1(\Omega)$ such that

$$\int_\Omega \frac{\partial u}{\partial x_1}(x)\frac{\partial v}{\partial x_1}(x) + \alpha(x)\frac{\partial u}{\partial x_2}(x)\frac{\partial v}{\partial x_2}(x) + \alpha(x)u(x)v(x)\,\mathrm{d}x = \int_\Omega 1 \cdot v(x)\,\mathrm{d}x \quad \forall v \in H^1(\Omega)\,,$$

[natural b.c] = Neumann b.c

where $\alpha \in C^0(\overline{\Omega})$ is a given coefficient function.

$-\operatorname{div}\left(\begin{bmatrix}1 & 0\\ 0 & \alpha\end{bmatrix}\operatorname{grad} u\right) + \alpha u$	$=$	1	in	Ω,
$\begin{bmatrix}1 & 0\\ 0 & \alpha\end{bmatrix}\operatorname{grad} u \cdot n$	$=$	0	on	$\partial\Omega$.

Alternative: $-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial}{\partial x_2}\left(\alpha \frac{\partial u}{\partial x_2}\right) + \alpha u$

$$= \int_\Omega \begin{bmatrix}1 & 0\\ 0 & \alpha(x)\end{bmatrix}\operatorname{grad} u \cdot \operatorname{grad} v + \alpha(x)\,uv\,dx$$

$\longrightarrow$ Section 1.8.

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(0-1.c) (4 pts.) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Give an alternative and simpler characterization of the following set of functions

$$\mathcal{A} := \left\{ \alpha \in C^0(\overline{\Omega}) : \ \exists \gamma > 0 \text{ such that } \ \mathsf{a}(\alpha; v, v) \geq \gamma \|v\|^2_{H^1(\Omega)} \quad \forall v \in H^1(\Omega) \right\},$$

with $\quad \mathsf{a}(\alpha; u, v) := \int_\Omega \frac{\partial u}{\partial x_1}(\boldsymbol{x}) \frac{\partial v}{\partial x_1}(\boldsymbol{x}) + \alpha(\boldsymbol{x}) \frac{\partial u}{\partial x_2}(\boldsymbol{x}) \frac{\partial v}{\partial x_2}(\boldsymbol{x}) + \alpha(\boldsymbol{x}) u(\boldsymbol{x}) v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$

for $u, v \in H^1(\Omega)$.

$$\mathcal{A} = \left\{ \alpha \in C^0(\overline{\Omega}) : \boxed{\alpha \text{ uniformly positive}} \right\}.$$

α positive, bounded away from 0

$$\|v\|^2_{H^1(\Omega)} = \int_\Omega \left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2 + v^2 \, dx$$

Alternative: $\exists \alpha_0 > 0 : \alpha(x) \geq \alpha_0 \ \forall x \in \overline{\Omega}$

Correct answer: $\alpha(x) > 0 \ \forall x \in \overline{\Omega}$

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Problem 0-2: Galerkin Matrices

Abstract Galerkin discretization of a linear variational problem leads a linear system of equations, whose system matrix is called a **Galerkin matrix**. This problem examines Galerkin matrices for typical bilinear forms connected with second-order elliptic BVPs from in an abstract setting and for a concrete finite-element trial and test space.

You need to be familiar with [Lecture → Section 2.2.3] and [Lecture → Section 2.4.5.2].
▷ problem name/problem code folder: GalerkinMatrices

Let $\Omega \subset \mathbb{R}^2$ be a regular hexagon in the plane. Throughout this problem we deal with the two bilinear forms

$$\mathsf{m}(u,v) := \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \qquad u,v \in L^2(\Omega)\,, \tag{0.2.1}$$

$$\mathsf{a}(u,v) := \int_\Omega \mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \qquad u,v \in H^1(\Omega)\,. \tag{0.2.2}$$

(0-2.a) ⚁ (4 pts.) Let $V_h \subset H^1(\Omega)$ be some *finite-dimensional* space equipped with some basis $\mathfrak{B} := \{b_h^1, \dots, b_h^N\}$, $N := \dim V_h$. We write $\mathbf{M} \in \mathbb{R}^{N,N}$ and $\mathbf{A} \in \mathbb{R}^{N,N}$ for the Galerkin matrices induced by m and a using $\mathfrak{B}$, that is,

$$\mathbf{M} = \left[\mathsf{m}(b_h^j, b_h^i)\right]_{i,j=1}^N \quad , \quad \mathbf{A} = \left[\mathsf{a}(b_h^j, b_h^i)\right]_{i,j=1}^N \,. \tag{0.2.3}$$

Decide, which properties of $\mathbf{A}$, $\mathbf{M}$, and $\mathbf{A}+\mathbf{M}$ hold for *any* choice of V_h and $\mathfrak{B}$.

matrix	symmetric	positive definite[a]	regular[b]	sparse[c]
$\mathbf{M}$	yes ☒	yes ☒	yes ☒	yes ○
	no ○	no ○	no ○	no ☒
$\mathbf{A}$	yes ☒	yes ○	yes ○	yes ○
	no ○	no ☒	no ☒	no ☒
$\mathbf{A}+\mathbf{M}$	yes ☒	yes ☒	yes ☒	yes ○
	no ○	no ○	no ○	no ☒

[a] A matrix $\mathbf{X} \in \mathbb{R}^{N,N}$ is called **positive definite**, if $\vec{\xi}^\top \mathbf{X}\vec{\xi} > 0$ for all $\vec{\xi} \in \mathbb{R}^N \setminus \{0\}$.
[b] A square matrix is called **regular**, when it is invertible.
[c] Here we call a matrix **sparse**, when it has a *fixed* small N-independent number of non-zero entries per row.

a, m are symmetric

$\mathsf{a}(x \mapsto 1, x \mapsto 1) = 0$!

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No b.c.

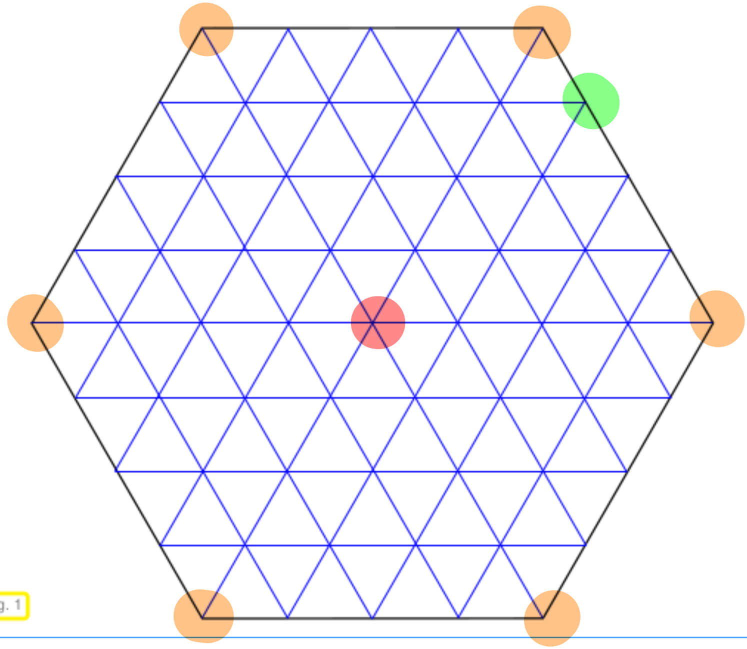

Fig. 1

In the sequel we consider finite-element Galerkin discretization based on the trial and test space $\mathcal{S}_1^0(\mathcal{M})$ of lowest-order Lagrangian finite element functions based on a mesh whose cells are all congruent equilateral triangles, see Fig. 1.

Throughout, we use the nodal basis of $\mathcal{S}_1^0(\mathcal{M})$ comprised of *tent functions*.

(0-2.b) (5 pts.) What are the dimensions of the following spaces

$$\dim \mathcal{S}_1^0(\mathcal{M}) = \boxed{\#\mathcal{V}(\mathcal{M})}\,,$$ [or node count]

$$\dim \mathcal{N}(\mathbf{M}) = \boxed{0}\,,$$

$$\dim \mathcal{N}(\mathbf{A}) = \boxed{1}\,?$$ [the constants!]

Here,

- $\mathbf{M}$ and $\mathbf{A}$ are the $\mathcal{S}_1^0(\mathcal{M})$-Galerkin matrices for the bilinear forms m and a, respectively.
- $\mathcal{N}(\mathbf{X})$ stands for the kernel/nullspace of a matrix $\mathbf{X}$.

Note. You may write your answer in terms of the number $\sharp\mathcal{V}(\mathcal{M})$ of vertices, the number $\sharp\mathcal{E}(\mathcal{M})$ of edges, and the number $\sharp\mathcal{M}$ of triangles of $\mathcal{M}$.

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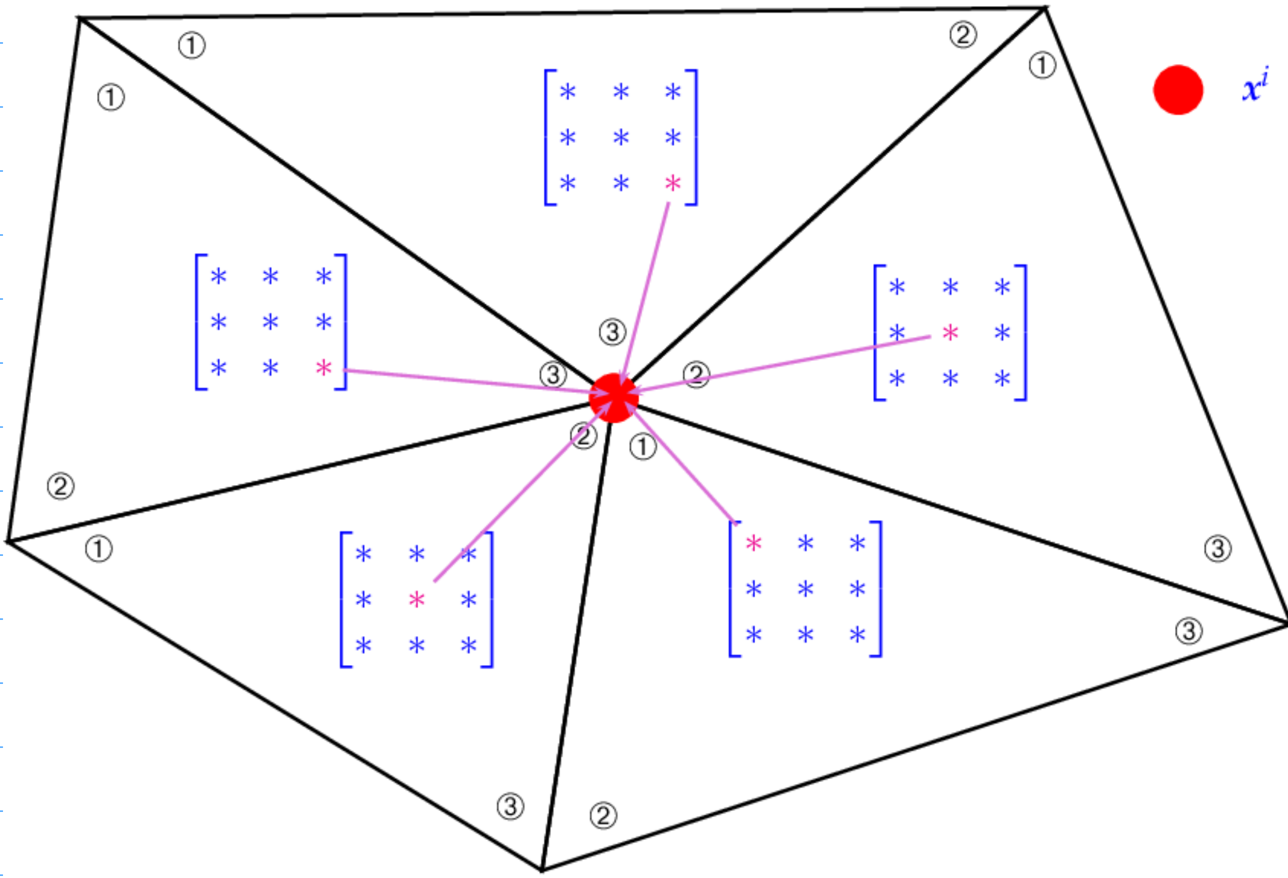
(0-2.c) (3 pts.) It is known that for our $\mathcal{S}_1^0(\mathcal{M})$-based finite element Galerkin discretization using the mesh sketched in Fig. 1 all **element matrices** for the bilinear form a are

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{3,3} .$$

Explicitly write down the set of diagonal entries of the Galerkin matrix $\mathbf{A}$:

$$\left\{ (\mathbf{A})_{i,i},\ i = 1, \dots, \dim \mathcal{S}_1^0(\mathcal{M}) \right\} = \left\{ \boxed{4,\ 6,\ 12} \right\} .$$



$(\mathbf{A})_{ii}$ by summing diagonal entries of element matrices of adjacent triangles

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Problem 0-3: Parametric FEM: Local Computations

In the case of parametric finite element methods the computation of element matrices can be organized efficiently by precomputing many expressions on the reference element(s), see [Lecture → § 2.8.3.6]. In this problem we recall the relevant formulas.

Assumes familiarity with [Lecture → § 2.8.3.6], [Lecture → § 2.8.3.14], [Lecture → § 2.8.3.15], and [Lecture → Eq. (2.8.3.13)].

▷ problem name/problem code folder: LocCompParamFE

As in [Lecture → Ex. 2.8.3.29], in this problem we consider the bilinear form

$$\mathsf{a}(u,v) := \int_{\Omega} \boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u(\boldsymbol{x}) \cdot \mathbf{grad}\,v(\boldsymbol{x}) + \gamma(\boldsymbol{x})u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \quad u,v \in H^1(\Omega)\,,$$

on a polygonally bounded computational domain $\Omega \subset \mathbb{R}^2$. Here, $\boldsymbol{\alpha} : \overline{\Omega} \to \mathbb{R}^{2,2}$ and $\gamma : \overline{\Omega} \to \mathbb{R}$ are uniformly positive (definite) coefficient functions.

(0-3.a) (8 pts.) In the context of *parametric* Lagrangian finite elements the following formula provides the entries of the **element matrix** [Lecture → Def. 2.7.4.5] for a mesh cell $K \in \mathcal{M}$:

$$(\mathbf{A}_K)_{ij} = \sum_{\ell=1}^{P} \boxed{\mathsf{A}} \Big(\boldsymbol{\alpha}(\boxed{\mathsf{B}})(\mathrm{D}\boldsymbol{\Phi}(\boxed{\mathsf{C}}))^{-\top}\,\mathbf{grad}\,\widehat{b}^i(\boxed{\mathsf{D}})) \cdot ((\mathrm{D}\boldsymbol{\Phi}(\boxed{\mathsf{E}}))^{-\top}\,\mathbf{grad}\,\boxed{\mathsf{F}})+ \gamma(\boxed{\mathsf{G}})\widehat{b}^j(\widehat{\zeta}_\ell)\widehat{b}^i(\widehat{\zeta}_\ell)\Big)\left|\boxed{\mathsf{H}}\right|\,, \quad i,j \in \{1,\dots,Q\}\,.$$

Put the appropriate expression into the placeholders by ticking the corresponding column:

expression	A	B	C	D	E	F	G	H
$\det \mathrm{D}\boldsymbol{\Phi}(\widehat{\zeta}_\ell)$	○	○	○	○	○	○	○	X
$\widehat{\omega}_\ell$	X	○	○	○	○	○	○	○
$\widehat{\zeta}_\ell$	○	○	X	X	X	○	○	○
$\widehat{b}^j(\widehat{\zeta}_\ell)$	○	○	○	○	○	X	○	○
$\boldsymbol{\Phi}(\widehat{\zeta}_\ell)$	○	X	○	○	○	○	X	○
$\mathrm{D}\boldsymbol{\Phi}(\widehat{\zeta}_\ell)$	○	○	○	○	○	○	○	○
$(\mathrm{D}\boldsymbol{\Phi}(\widehat{\zeta}_\ell))^{\top}$	○	○	○	○	○	○	○	○

Symbols:

- $\widehat{\omega}_\ell \in \mathbb{R}$, $\ell \in \{1,\dots,P\}$, $P \in \mathbb{N}$ $\hat{=}$ quadrature weights of local quadrature rule on $\widehat{K}$,
- $\widehat{\zeta}_\ell$, $\ell \in \{1,\dots,P\}$ $\hat{=}$ quadrature nodes in reference element $\widehat{K}$,
- $\boldsymbol{\Phi} : \widehat{K} \to K$ bijective mapping from reference element $\widehat{K}$ to K,
- $\widehat{b}^j$, $j \in \{1,\dots,Q\}$, $Q \in \mathbb{N}$, local shape functions on reference element $\widehat{K}$.

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(0-3.b) (4 pts.) The **lf::geometry::Geometry** and **lf::quad::QuadRule** classes of LEHRFEM++ provide a number of member functions to support the local computations for parametric Lagrangian finite elements.

In the table below Match the member functions and expressions that play a role in local computations in the context of parametric Lagrangian finite elements: the member function is to offer direct access to the expression (We use the symbols introduced in Sub-problem (0-3.a)).

Member function	$\widehat{\zeta}_\ell$	$\mathbf{\Phi}(\widehat{\zeta})$	$\mathrm{D}\mathbf{\Phi}(\widehat{\zeta})^{-\top}$	$\det \mathrm{D}\mathbf{\Phi}(\widehat{\zeta})$
`Geometry::Global()`	○	X	○	○
`lf::quad::QuadRule::Weights()`	○	○	○	○
`Geometry::IntegrationElement()`	○	○	○	X
`lf::quad::QuadRule::Points()`	X	○	○	○
`Geometry::JacobianInverseGramian()`	○	○	X	○

Here `Geometry` stands for `lf::geometry::Geometry`.

→ realizes $\phi : \hat{K} \longrightarrow K$