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ETH Lecture 401-0674-00L Numerical Methods for (Partial) Differential Equations

Numerical Methods for (Partial) Differential Equations

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Welcome & Introduction

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It is highly desirable that questions are *submitted* at least a few hours *before* the start of the Q&A session so that the lecturer has the opportunity to structure his or her answer. Submission of questions should be done through dedicated DISCUNA *chat channels*. A separate channel has been set up for each week in which a regular Q&A session will take place.

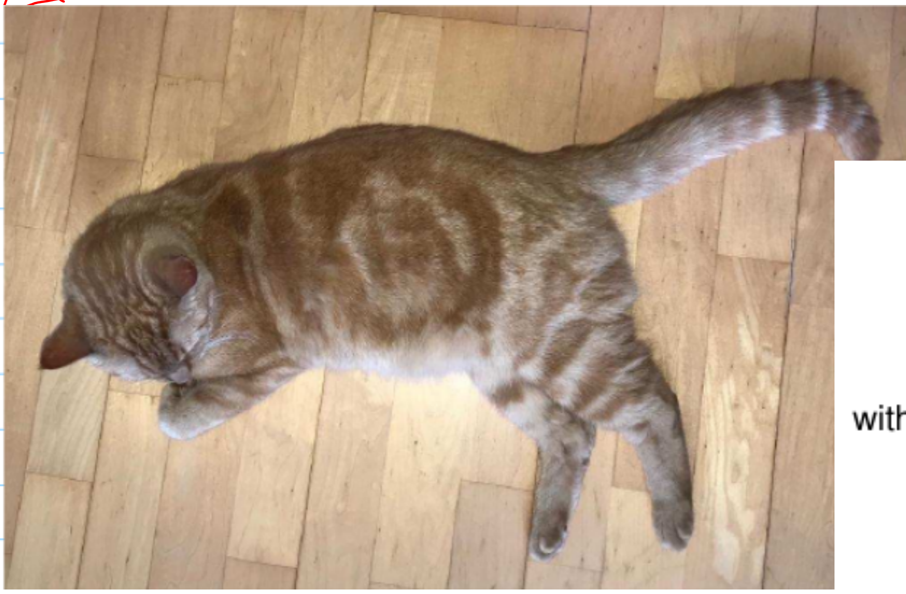
Important links and resources:

(1) Course directory page:
https://www.vorlesungen.ethz.ch/Vorlesungsverzeichnis/lerneinheit.view?lerneinheitId=188112&semkez=2025S&ansicht=LEHRVERANSTALTUNGEN&lang=de
(2) Moodle page: https://moodle-app2.let.ethz.ch/course/view.php?id=24006
(3) Lecture document: https://www.sam.math.ethz.ch/~grsam/NUMPDEFL/NUMPDE.pdf
(4) Homework problem collection:
https://people.math.ethz.ch/~grsam/NUMPDEFL/HOMEWORK/NPDEFL_Problems.pdf
(5) Course code repository: https://gitlab.math.ethz.ch/ralfh/NPDERepo
(6) Course repository: https://people.math.ethz.ch/~grsam/NUMPDEFL/
(7) Course polybox folder:
https://polybox.ethz.ch/index.php/s/Mn25THrx1yjvcPw ,
PW: NPDE25 (contains course videos and tablet notes)
(8) Discuna join link: https://app.discuna.com/invite/9QLX7oxbfNl22FceUwEM

(2)

$$\text{PDE}: \quad F(u, Du) = 0$$

$\hookrightarrow$ function $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$



Reaction-diffusion PDE $\triangleright$ Chapter 5

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= \gamma f(u,v)\,, \\ \frac{\partial v}{\partial t} - \Delta v &= \gamma g(u,v) \end{aligned} \qquad \text{in} \quad \Omega\,,$$

with functions

$$\begin{aligned} f(u,.v) &:= a - u - h(u,v)\,, \\ g(u,v) &:= \alpha(b - v) - h(u,v)\,, \\ h(u,v) &:= \frac{uv}{1 + u + Ku^2}\,, \end{aligned}$$

(Stationary, incompressible) Navier-Stokes equations: $\triangleq$ a model for fluid flow

$$\begin{aligned} -\nu\Delta\mathbf{u} + D\mathbf{u}\cdot\mathbf{u} + \mathbf{grad}\,p &= \mathbf{f} \quad \text{in } \Omega\,, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega\,, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{0.4.3.1}$$

Laplacian

ν	$\hat{=}$	dynamic viscosity
$\mathbf{f} : \Omega \mapsto \mathbb{R}^3$	$\hat{=}$	given external force field
$\mathbf{u} : \Omega \mapsto \mathbb{R}^3$	$\hat{=}$	velocity field (unknown)
$p : \Omega \mapsto \mathbb{R}$	$\hat{=}$	pressure (unknown)

PDE $\neq$ "just an equation"

It is always a model of a (physical/technical/economics/social) system behind.

③

Better title for this course:

Numerical Methods for Continuum models with local interactions

State variables are functions/fields $\underline{u}: \Omega \subset \mathbb{R}^d \to \mathbb{R}^n$

$\longrightarrow$ derivative-based

model encodes structure $\Leftarrow$ embedded in a (physical) context

$\downarrow$

should be preserved by numerical method

$\mathrm{div}\, \underline{u} = 0 \longrightarrow$ incompressibility [$\to$ Chapter 13]

[$\mathrm{div}\, \underline{u} = \nabla \cdot \underline{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$

$\underline{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$

Gauss theorem: $V \subset \mathbb{R}^d$ $\quad \int_V \mathrm{div}\, \underline{u}\, d\underline{x} = \int_{\partial V} \underline{u} \cdot \underline{n}\, dS$

1D: $\displaystyle\int_a^b \frac{\partial u}{\partial x}\, dx = u(b) - u(a)$

$\mathrm{grad}\, p = \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix}, \quad p: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$]

Contents [organized by model classes]

I. Equilibrium model

↳ Elastic membranes, electrostatics, ...

Ch. 1 : theory
Ch. 2 : Finite-element methods (FEM)
Ch. 3 : theory of FEM
Ch. 4 : beyond FEM
Ch. 5 : non-linear models

II. Evolution models

↳ Heat equation, wave propagation

Ch. 9 : 2nd-order linear evolution problems

III. Models w/ special structure

Ch. 12 : finite-element exterior calculus

Ch. 13 : Fluid equation

Focus :
- Derivation on methods
- Scope of methods
- on algorithms → C++ coding

Organization

→ flipped classroom

A flipped-classroom course

This course will follow the **flipped-classroom** paradigm:

Learning by **self-study** guided by

instruction videos tablet notes	lecture notes	interactive Q&A sessions	homework tutorial classes
learning unit (assigned weekly)		Please submit questions	7-8h/week

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