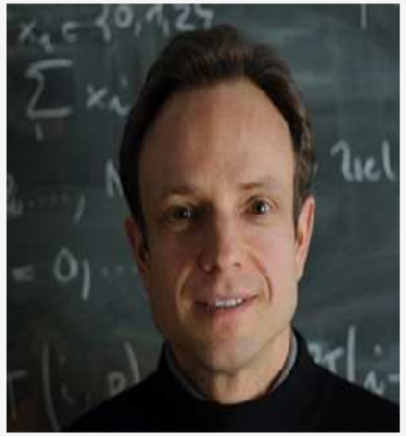
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 6-4: The One-Dimensional Wave Equation with Absorbing Boundary Condition

Prof. R. Hiptmair, SAM, ETH Zurich

Date: April 19, 2019



1D wave propagation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{on }]0,1[\times]0,T[\;. \tag{6.4.1}$$

$c \mathrel{\hat=}$ wave speed

$$c\frac{\partial u}{\partial x}(0,t) - \frac{\partial u}{\partial t}(0,t) = 0\,, \tag{6.4.2a}$$

absorbing b.c.

$$u(1,t) = g(t) := \begin{cases} \sin t & 0 \le t \le \pi, \\ 0 & \text{otherwise}, \end{cases} \tag{6.4.2b}$$

spatial Dirichlet b.c.

$$\underbrace{u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < 1\,.}_{\text{initial conditions}} \tag{6.4.2c}$$

a) Spatial variational formulation

Test space $V_0 := \{v \in H^1(\Omega) : v(1) = 0\}$

Test with $v \in V_0$ & integrate over $]0,1[$ & i.b.p.

$$\int_0^1 \frac{\partial^2 u}{\partial t^2}(x,t)\,v(x)\,\mathrm{d}x + c\int_0^1 \frac{\partial u}{\partial x}(x,t)\frac{\partial v}{\partial x}(x)\,\mathrm{d}x + c\frac{\partial u}{\partial x}(0,t)\,v(0) = 0 \quad \forall v \in V_0$$

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$u :]0,T[\;\rightarrow\; \{v \in H^1(]0,1[) : v(1) = g(t)\}$

② ▷ t-dependent affine trial space

$$\mathsf{m}(\ddot{u},v)+\mathsf{b}(\dot{u},v)+\mathsf{a}(u,v)=0\quad\forall v\in V_0\,, \tag{6.4.3}$$

$$\text{where}\quad \mathsf{m}(u,v):=\int_0^1 uv\,\mathrm{d}x\,, \tag{6.4.4}$$

$$\mathsf{b}(u,v):=u(0)v(0)\,,$$

$$\mathsf{a}(u,v):=c\int_0^1\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x\,.$$

b) Spatial Galerkin discretization based $S_1^0(\mathcal{M})$, $\mathcal{M}\hat{=}$ equidistant mesh

By the method-of-lines approach as outlined above we arrive at an ordinary differential equation of the form

$$\mathbf{M}\ddot{\vec{\mu}}+\mathbf{B}\dot{\vec{\mu}}+\mathbf{A}\vec{\mu}=\vec{\varphi}(t)\,, \tag{6.4.5}$$

with matrices $\mathbf{M},\mathbf{A},\mathbf{B}\in\mathbb{R}^{N,N}$ and a time-dependent vector $\vec{\varphi}$. Here N stands for the dimension of the finite element test space.
Give concrete formulas for the entries of $\mathbf{M},\mathbf{A},\mathbf{B}\in\mathbb{R}^{N,N}$ and the components of $\vec{\varphi}$.

▷ All integrations based on trapezoidal rule
⇒ Mass M matrix diagonal

$$(\mathbf{B})_{ij}=\begin{cases}1 & \text{for } i=j=1\\ 0 & \text{else}\end{cases}$$

Galerkin matrices for full FE space $S_1^0(\mathcal{M})$

$$\mathbf{M}_{\text{full}}=h\begin{pmatrix}\frac12 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac12\end{pmatrix},\qquad \mathbf{A}_{\text{full}}=\frac{c}{h}\begin{pmatrix}1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1\\ & & & -1 & 1\end{pmatrix}.$$

Non-zero Dirichlet b.c. are taken into account by offset function

$$u_h(t)=\sum_{\ell=1}^{N}\mu_\ell(t)\,b_h^\ell+g(t)\,b_h^{N+1}$$

$N:=\dim(S_1^0(\mathcal{M})\cap V_0)$ — tent function at $x=1$

We start from:

$$\mathsf{m}(\ddot{u}_h(t),b_h^\ell)+\mathsf{b}(\dot{u}_h(t),b_h^\ell)+\mathsf{a}(u_h(t),b_h^\ell)=0,\quad \ell=1,\dots,N$$

$$(\vec{\varphi})_\ell=\cancel{-\mathsf{m}(\ddot{g}(t)b_h^{N+1},b_h^\ell)}\cancel{-\mathsf{b}(\dot{g}(t)b_h^{N+1},b_h^\ell)}-\mathsf{a}(g(t)b_h^{N+1},b_h^\ell),\quad \ell=1,\dots,N$$

$$=\begin{cases}\frac{c}{h}g(t) & \text{for } \ell=N\\ 0 & \text{elsewhere}\end{cases}$$

③ Leapfrog timestepping : $\vec{v}^{(j)} \approx \dot{\vec{\mu}}(t_{j+1/2})$

$$\mathbf{M}\frac{\vec{v}^{(j+1/2)}-\vec{v}^{(j-1/2)}}{\tau} = -\mathbf{A}\mu^{(j)} - \mathbf{B}\underbrace{\frac{\vec{v}^{(j+1/2)}+\vec{v}^{(j-1/2)}}{2}}_{\approx \dot{\vec{\mu}}(t_j)} + \varphi(t_j)\;, \qquad (6.4.6)$$
$$\frac{\vec{\mu}^{(j+1)}-\vec{\mu}^{(j)}}{\tau} = \vec{v}^{(j+1/2)}\;.$$

Initial step : $\vec{v}^{(-1/2)} = 2\vec{v}_0 - \vec{v}^{(1/2)}\;.$ (6.4.7)

c) Implementation of leapfrog timestepping

Initial step: (6.4.6) for $j=0$: $\mapsto v^{(1/2)} = 0$

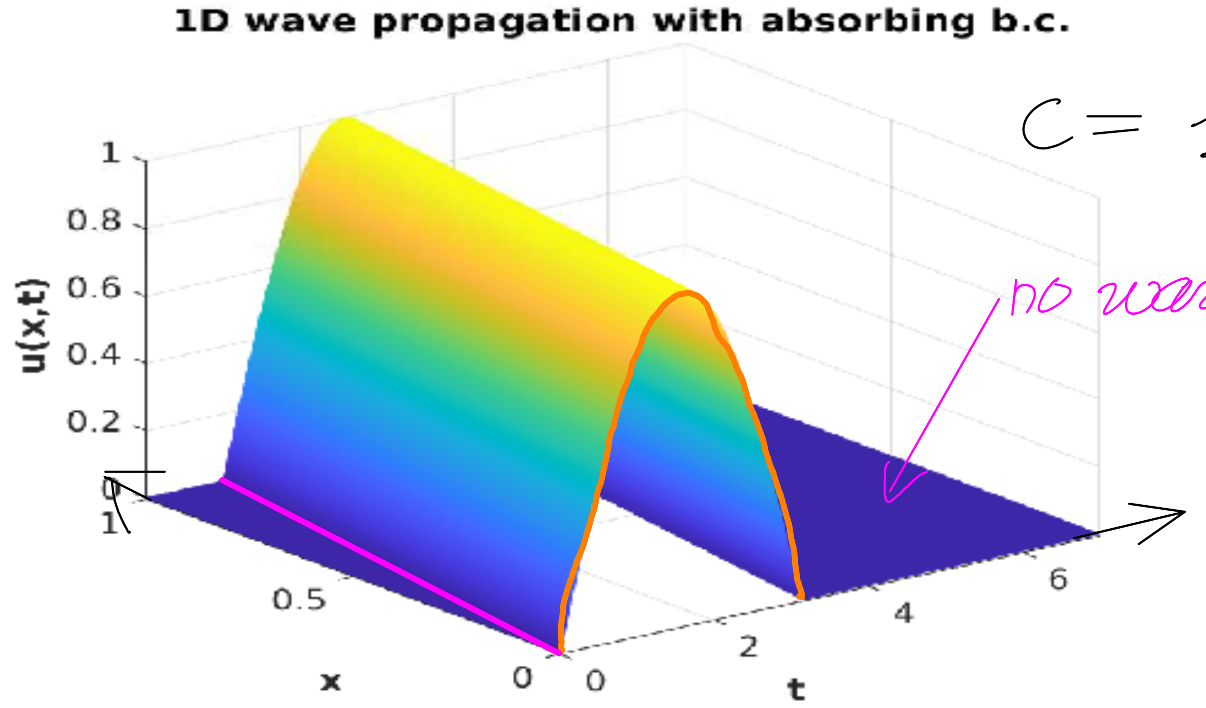
$$2\mathbf{M}\vec{v}^{(1/2)} = -\tau\mathbf{A}\vec{\mu}_0 - \tau\mathbf{B}\vec{v}_0 + 2\mathbf{M}\vec{v}_0 + \tau\vec{\varphi}(0)\;. = 0 \qquad (6.4.9)$$

initial values $= 0$

$$\vec{\mu}^{(j+1)} := \vec{\mu}^{(j)} + \tau\vec{v}^{(j+1/2)}\;,\quad j \in \mathbb{N}_0\;, \qquad (6.4.10)$$
$$(\mathbf{M}+\tfrac{1}{2}\tau\mathbf{B})\vec{v}^{(j+1/2)} := -\tau\mathbf{A}\vec{\mu}^{(j)} - \tfrac{1}{2}\mathbf{B}\vec{v}^{(j-1/2)} + \mathbf{M}\vec{v}^{(j-1/2)} + \tau\vec{\varphi}(t_j)\;,\quad j \in \mathbb{N}\;. \qquad (6.4.11)$$

diagonal matrix

d)



$c = 1$

no wave

Excitation just propagates, cf. d'Alembert solution

e) In the file `1dwaveabsorbingbc.cc` write a C++ function

```
std::pair<Eigen::VectorXd, Eigen::VectorXd> computeEnergies(
    const Eigen::MatrixXd &full_solution,
    double c, double tau);
```

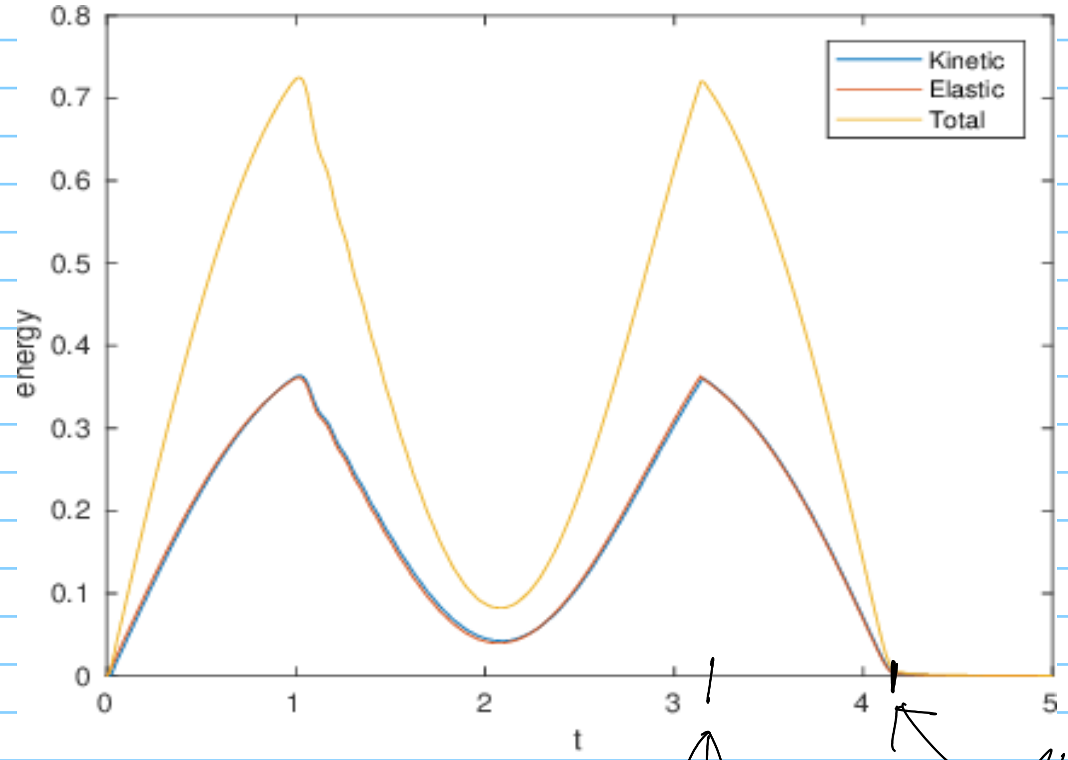
that accepts as an argumment the output matrix of `waveLeapfrogABC()` from Sub-problem (6-4.c) plus the constant c and the timestep size $\tau > 0$, and computes the discrete energies (using the notations of (6.4.6))

potential energy: $E_{\text{pot}}^{(j)} := \frac{1}{2}\vec{\mu}^{(j)}\mathbf{A}\vec{\mu}^{(j)}\;,\qquad j = 0,\dots,m\;,$

kinetic energy: $E_{\text{kin}}^{(j)} := \frac{1}{2}\vec{v}^{(j+\frac{1}{2})}\mathbf{M}\vec{v}^{(j+\frac{1}{2})}\;,\qquad j = 0,\dots,m-1\;.$

These energies are returned as a pair of vectors, the potential energy in the first component, the kinetic energy in the second.

(4) $\vec{\nu}^{(j+1/2)} := \frac{1}{\tau}\left(\vec{\mu}^{(j)} - \vec{\mu}^{(j+1)}\right)$



$c = 1$

excitation stops

All the waves have left domain at time $\pi + 1$

h) Now we consider the method-of-lines ode (6.4.5) without excitation, that is, in the case $\vec{\varphi} \equiv 0$. Show that even in this case he total energy

$$E(t) = \frac{1}{2}\dot{\vec{\mu}}(t)^\top \mathbf{M}\dot{\vec{\mu}}(t) + \frac{1}{2}\vec{\mu}(t)^\top A\vec{\mu}(t) \tag{6.4.15}$$

will not be conserved, but decay, that is, $t \mapsto E(t)$ is decreasing, if $\vec{\mu}(t)$ solves (6.4.5).

Hint: Generalized product rule

$S = S^T$: $\frac{d}{dt}\left(\vec{\mu}(t)^T S \vec{\mu}(t)\right) = 2\dot{\vec{\mu}}(t)^T S \vec{\mu}(t)$

$\triangleright$ $\frac{dE}{dt}(t) = \dot{\vec{\mu}}(t)^\top \mathbf{M}\ddot{\vec{\mu}}(t) + \dot{\vec{\mu}}(t)^\top A\vec{\mu}(t)$.

MOL-ODE : $M\ddot{\vec{\mu}} + B\dot{\vec{\mu}} + A\vec{\mu} = 0$

$\cdot\dot{\vec{\mu}}^T$: $\dot{\vec{\mu}}(t)^\top \mathbf{M}\ddot{\vec{\mu}}(t) + \dot{\vec{\mu}}(t)^\top A\vec{\mu}(t) + \dot{\vec{\mu}}(t)^\top \mathbf{B}\dot{\vec{\mu}}(t) = 0$.

$\triangleright$ $\frac{dE}{dt}(t) = -\dot{\vec{\mu}}(t)^\top \mathbf{B}\dot{\vec{\mu}}(t) \leq 0$

$\dot{\vec{\mu}}(0) \neq 0 \Rightarrow \frac{dE}{dt}(t) < 0$