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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial
Problem 1-8: A coupled reaction-diffusion problem

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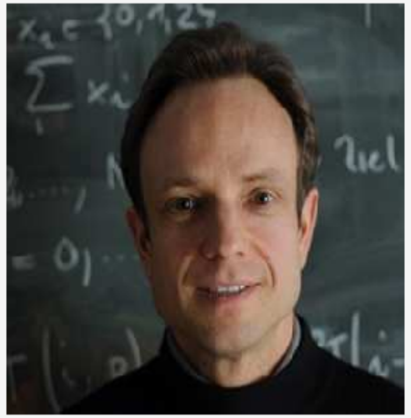

$$\begin{aligned} -\Delta u &= c(x)w + f \\ -\Delta w &= -u \end{aligned} \quad \text{in } \Omega \;, \quad \begin{aligned} \mathbf{grad}\, u \cdot \boldsymbol{n} &= 0 \\ \mathbf{grad}\, w \cdot \boldsymbol{n} &= -u \end{aligned} \quad \text{on } \partial\Omega \;. \tag{1.8.1}$$

$u, w : \Omega \to \mathbb{R}$, $f \in L^2(\Omega)$

a, Variational formulation

$$\begin{aligned} -\Delta u &= c(x)w + f \\ -\Delta w &= -u \end{aligned} \quad \text{in } \Omega \;, \quad \begin{aligned} \mathbf{grad}\, u \cdot \boldsymbol{n} &= 0 \\ \mathbf{grad}\, w \cdot \boldsymbol{n} &= -u \end{aligned} \quad \text{on } \partial\Omega \;. \tag{1.8.1}$$

$$\int_\Omega \mathrm{grad}\, u \cdot \mathrm{grad}\, p \, dx = \int_\Omega (c(x)w + f) p \, dx \quad \forall p \in H^1(\Omega)$$

$$\int_\Omega \mathrm{grad}\, w \cdot \mathrm{grad}\, q \, dx = \int_\Omega -u q \, dx \quad \forall q \in H^1(\Omega)$$

b, Variational formulation in standard form:

$$\tilde{u} \in V_0 \quad a(\tilde{u}, \tilde{v}) = \ell(\tilde{v}) \quad \forall \tilde{v} \in V_0$$

$$\begin{aligned} \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, p \, \mathrm{d}x \quad - \quad \int_\Omega c(x) w \, p \, \mathrm{d}x \quad &= \int_\Omega f \, p \, \mathrm{d}x \quad \forall p \in H^1(\Omega) \,, \\ \int_\Omega u \, q \, \mathrm{d}x + \int_{\partial\Omega} u \, q \, \mathrm{d}S \; + \; \int_\Omega \mathbf{grad}\, w \cdot \mathbf{grad}\, q \, \mathrm{d}x \quad &= 0 \quad \forall q \in H^1(\Omega) \,. \end{aligned} \tag{1.8.4}$$

Trick: Just add the two equations!

$$\int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, p \, \mathrm{d}x - \int_\Omega c(x) w \, p \, \mathrm{d}x + \int_\Omega u \, q \, \mathrm{d}x + \int_{\partial\Omega} u \, q \, \mathrm{d}S + \int_\Omega \mathbf{grad}\, w \cdot \mathbf{grad}\, q \, \mathrm{d}x = \int_\Omega f \, p \, \mathrm{d}x \quad \forall (p,q) \in V_0 \,. \tag{1.8.5}$$

Remember from LA: Algebraic product of vector space

$$V_0 = H^1(\Omega) \times H^1(\Omega)$$

$$\mathsf{a}((u,w),(p,q)) = \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, p \,\mathrm{d}\mathbf{x} - \int_\Omega c(\mathbf{x}) w\, p \,\mathrm{d}\mathbf{x} + \int_\Omega u\, q \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} u\, q \,\mathrm{d}S + \int_\Omega \mathbf{grad}\, w \cdot \mathbf{grad}\, q \,\mathrm{d}\mathbf{x}\,, \tag{1.8.6}$$

$$\ell((p,q)) = \int_\Omega f\, p \,\mathrm{d}\mathbf{x}\,. \tag{1.8.7}$$

c) Uniqueness of solutions ?

Remember: Solution of LSE $A\vec{x} = \vec{b}$ unique

$\Longleftrightarrow$ $\mathrm{Ker}(A) = \{0\}$

Lemma 1.8.8. Uniquness of solutions of linear variational problems

Assuming they exist, solutions of a linear variational problem [Lecture → Def. 1.4.1.6]

$$u \in V_0: \quad \mathsf{a}(u,v) = \ell(v) \quad \forall v \in V_0\,,$$

are unique, if and only if

$$\mathsf{a}(u,v) = 0 \quad \forall v \in V_0 \iff u = 0\,.$$

Examine: $a((u,w),(p,q)) = 0 \quad \forall (p,q) \in V_0$

Idea: Choose special test functions depending on u & w:

- Spawn terms "with a sign"
- Exploit cancellation

$$\mathsf{a}((u,w),(p,q)) = \int_\Omega \cancel{\mathbf{grad}\, u \cdot \mathbf{grad}\, p} \,\mathrm{d}\mathbf{x} + \int_\Omega c(\mathbf{x}) w\, w \,\mathrm{d}\mathbf{x} + \int_\Omega u\, u \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} u\, u \,\mathrm{d}S + \int_\Omega \cancel{\mathbf{grad}\, w \cdot \mathbf{grad}\, q} \,\mathrm{d}\mathbf{x}\,, = 0 \quad \forall (p,q) \tag{1.8.6}$$

$$\ell((p,q)) = \int_\Omega f\, p \,\mathrm{d}\mathbf{x}\,. \tag{1.8.7}$$

$p \to -w$, $q \to u$

$$\mathsf{a}((u,w),(-w,u)) = \int_\Omega c(\mathbf{x}) w^2 + u^2 \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} u^2 \,\mathrm{d}S = 0\,,$$

uniformly positive

$\triangleright$ $u = w = 0$

d) $c \equiv 1$. a s.p.d ? Guess: No

Idea: Find $(u,w) \in V_0$ s.t. $a((u,w),(u,w)) < 0$

Hint: Choose constants!

$$\mathsf{a}((\underbrace{\{\mathbf{x} \to 1\}}_{u}, \underbrace{\{\mathbf{x} \to -1\}}_{w}), (\{\mathbf{x} \to 1\}, \{\mathbf{x} \to -1\})) = -\int_\Omega 1 \,\mathrm{d}\mathbf{x} + \int_\Omega 1 \,\mathrm{d}\mathbf{x} - \int_{\partial\Omega} 1 \,\mathrm{d}S = -1\,.$$