ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 6-3: Decaying Method-of-Lines Solution with Implicit-Euler Timestepping

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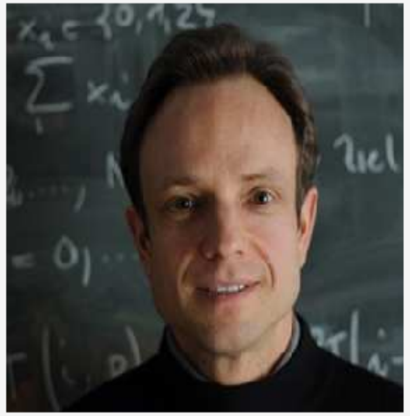

Abstract parabolic evolution problem

s.p.d. on vector space V_0

no excitation

$$\begin{aligned} \mathsf{m}(\dot{u},v) + \mathsf{a}(u,v) &= 0 \quad \forall v \in V_0, \quad 0<t<T, \\ u(0) &= u_0 \in V_0 \,, \end{aligned} \tag{6.3.1}$$

[related → Sect 6.2.3] a, → e,

General P.F.-inequ.:

$\exists \gamma > 0: \ \mathsf{a}(v,v) \geq \gamma\, \mathsf{m}(v,v) \quad \forall v \in V_0$

Lemma 6.2.3.3. Decay of solutions of parabolic evolutions

For $f \equiv 0$ the solution $u(t)$ of (7.1.2.4) satisfies

$$\|u(t)\|_m \leq e^{-\gamma t}\|u_0\|_m \quad , \quad \|u(t)\|_a \leq e^{-\gamma t}\|u_0\|_a \quad \forall t \in]0,T[\,,$$

where $\gamma > 0$ is the constant from (7.1.3.1), and $\|\cdot\|_a$, $\|\cdot\|_m$ stand for the energy norms induced by $\mathsf{a}(\cdot,\cdot)$ and $\mathsf{m}(\cdot,\cdot)$, respectively.

MOL-approach: $V_0 \to V_{0,h} \Rightarrow$ ODE

$$\begin{aligned} \mathbf{M}\left\{\frac{d}{dt}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu} &= 0, \quad 0<t<T, \\ \vec{\mu}(0) &= \vec{\mu}_0 \,, \end{aligned} \tag{6.3.3}$$

s.p.d.

② a) Implicit Euler (timestep $\tau > 0$)

$$\frac{d}{dt}\vec{\mu}(t) = \frac{\vec{\mu}^{(j)} - \vec{\mu}^{(j-1)}}{\tau}$$

$$\Rightarrow \quad M(\vec{\mu}^{(j)} - \vec{\mu}^{(j-1)}) = -\tau A \vec{\mu}^{(j)}$$

c) Diagonalization:

$$AT = MTD, \quad D = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$$

$$T^T M T = I$$

Transformed variable: $\vec{\eta}^{(j)} := T^T M \vec{\mu}^{(j)} = T^{-1}\vec{\mu}^{(j)}$

$$T^T M(\vec{\mu}^{(j)} - \vec{\mu}^{(j-1)}) = -\tau \underbrace{T^T A T}_{=MTD} T^{-1}\vec{\mu}^{(j)}$$

$$\downarrow$$

$$\vec{\eta}^{(j)} - \vec{\eta}^{(j-1)} = -\tau D \vec{\eta}^{(j)}$$

$$\eta_i^{(j)} - \eta_i^{(j-1)} = -\tau \lambda_i \eta_i^{(j)}$$

d) The s.p.d. matrices $\mathbf{A} \in \mathbb{R}^{N,N}$ and $\mathbf{M} \in \mathbb{R}^{N,N}$ define norms on $\mathbb{R}^N$:

$$\left\|\vec{\xi}\right\|_{\mathbf{M}} := (\vec{\xi}^\top \mathbf{M}\vec{\xi})^{\frac{1}{2}} \;, \quad \left\|\vec{\xi}\right\|_{\mathbf{A}} := (\vec{\xi}^\top \mathbf{A}\vec{\xi})^{\frac{1}{2}} \,. \tag{6.3.4}$$

What will the norms $\|\cdot\|_{\mathbf{A}}$ and $\|\cdot\|_{\mathbf{M}}$ of a coefficient vector $\vec{\mu}$ become for the transformed vector $\vec{\eta} := \mathbf{T}^\top \mathbf{M}\vec{\mu}$?

$$\|\vec{\mu}\|_M^2 = \vec{\mu}^T M \vec{\mu} = \vec{\eta}^T \underbrace{T^T M T}_{=Id} \vec{\eta} = \|\vec{\eta}\|^2$$

$$\|\vec{\mu}\|_A^2 = \vec{\mu}^T A \vec{\mu} = \vec{\eta}^T \underbrace{T^T A T}_{MTD} \vec{\eta} = \vec{\eta}^T D \vec{\eta}$$

e) Express $\mathsf{m}(u_h, u_h)$ and $\mathsf{a}(u_h, u_h)$ through the coefficient vector $\vec{\mu}$ associated with a function $u_h \in V_{0,h}$ from the discrete Galerkin trial/test space and through the Galerkin matrices $\mathbf{A}$ and $\mathbf{M}$. How does (6.3.2) read when stated in terms of matrices and coefficient vectors? What is the relationship of γ and the generalized eigenvalues λ_i?

$$\mathsf{m}(u_h, u_h) = \mathsf{m}\left(\sum_{i=1}^N \mu_i b_h^i, \sum_{j=1}^N \mu_j b_h^j\right) = \sum_{i=1}^N \sum_{j=1}^N \mu_i \mu_j \mathsf{m}(b_h^i, b_h^j) = \|\vec{\mu}\|_{\mathbf{M}}^2$$

$$\mathsf{a}(u_h, u_h) = \mathsf{a}\left(\sum_{i=1}^N \mu_i b_h^i, \sum_{j=1}^N \mu_j b_h^j\right) = \sum_{i=1}^N \sum_{j=1}^N \mu_i \mu_j \mathsf{a}(b_h^i, b_h^j) = \|\vec{\mu}\|_{\mathbf{A}}^2$$

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$$\exists \gamma > 0: \quad a(v_h, v_h) \geq \gamma m(v_h, v_h) \quad \forall v \in V_{0,h} \tag{6.3.2}$$

$$\Downarrow \quad v_h \leftrightarrow \vec{\nu} \in \mathbb{R}^N$$

$$\vec{\nu}^T A \vec{\nu} \geq \gamma \vec{\nu}^T M \vec{\nu} \quad \forall \vec{\nu}$$

$$\Downarrow \quad \vec{\eta} := T^{-1}\vec{\nu}$$

$$\vec{\eta}^T D \vec{\eta} \geq \gamma \vec{\eta}^T \vec{\eta} \Leftrightarrow \eta_i^2 \lambda_i \geq \gamma \eta_i^2 \quad \forall \eta_i$$

$$\Leftrightarrow \quad \lambda_i \geq \gamma$$

f) Uniform timestep $\tau > 0$

In the sequel we assume a uniform timestep $\tau > 0$. Show that

$$\left\|\vec{\mu}^{(j)}\right\|_{\mathbf{M}} \leq \frac{1}{1+\gamma\tau}\left\|\vec{\mu}^{(j-1)}\right\|_{\mathbf{M}}, \tag{6.3.5}$$

where $\gamma > 0$ is the constant from (6.3.2).

By diagonalization: $\vec{\eta}^{(j)} = (I + \tau D)^{-1}\vec{\eta}^{(j-1)}$

$$\|\vec{\eta}^{(j)}\| = \|(I+\tau D)^{-1}\vec{\eta}^{(j-1)}\|$$

$$\leq \max_i \frac{1}{1+\tau\lambda_i} \cdot \|\eta^{(j-1)}\|$$

$$\leq \frac{1}{1+\tau\gamma}\|\vec{\eta}^{(j-1)}\|$$

g) In the sequel we assume a uniform timestep $\tau > 0$. Show that

$$\left\|\vec{\mu}^{(j)}\right\|_{\cancel{\mathbf{M}}\,A} \leq \frac{1}{1+\gamma\tau}\left\|\vec{\mu}^{(j-1)}\right\|_{\cancel{\mathbf{M}}\,A}, \tag{6.3.5}$$

where $\gamma > 0$ is the constant from (6.3.2).

By diagonalization:

$$\|\vec{\nu}^{(j)}\|_A = \|\sqrt{D}\,\vec{\eta}^{(j)}\| = \|\sqrt{D}(I+\tau D)^{-1}\vec{\eta}^{(j-1)}\|$$

$$\leq \frac{1}{1+\tau\gamma}\|\sqrt{D}\,\vec{\eta}^{(j-1)}\| = \frac{1}{1+\tau\gamma}\|\vec{\nu}^{(j-1)}\|_A$$

h) Relate time-discrete & continuous decay results:

In the sequel we assume a uniform timestep $\tau > 0$. Show that

$$\left\|\vec{\mu}^{(j)}\right\|_{\mathbf{M}} \leq \frac{1}{1+\gamma\tau}\left\|\vec{\mu}^{(j-1)}\right\|_{\mathbf{M}}, \tag{6.3.5}$$

$\tau > 0$: timestep

where $\gamma > 0$ is the constant from (6.3.2).

$\Updownarrow$

semi-discrete

Lemma 6.3.7. Decay of solutions of parabolic evolutions

Assuming (6.3.2)*, the solution* $t \mapsto \vec{\mu}(t)$ *of the method-of-lines ODE* (6.3.3) *satisfies*

$$\|\vec{\mu}(t)\|_{\mathbf{M}} \leq e^{-\gamma t}\|\vec{\mu}(0)\|_{\mathbf{M}} \quad , \quad \|\vec{\mu}(t)\|_{\mathbf{A}} \leq e^{-\gamma t}\|\vec{\mu}(0)\|_{\mathbf{A}} \quad \forall t \geq 0 ,$$

with $\|\cdot\|_{\mathbf{A}}$ *and* $\|\cdot\|_{\mathbf{M}}$ *defined in* (6.3.4).

$$(6.3.5) \triangleright \quad \left\|\vec{\mu}^{(j)}\right\|_{\mathbf{M}} \leq \frac{1}{1+\gamma\tau}\left\|\vec{\mu}^{(j-1)}\right\|_{\mathbf{M}} = \left(\frac{1}{1+\gamma\tau}\right)^{j}\left\|\vec{\mu}^{(0)}\right\|_{\mathbf{M}} .$$

Fix $t \geq 0$: $\vec{\mu}(t) \approx \vec{\mu}^{(j)}$, $j \approx t/\tau$

$\tau \approx t/j$

For $\tau \to 0 \Rightarrow j \to \infty$

$$\|\vec{\mu}^{(j)}\|_M \leq \left(\frac{1}{1+\gamma t/j}\right)^j \|\vec{\mu}^{(0)}\|_M$$

$$= \left(1 - \frac{\gamma t}{j+\cancel{\gamma t}}\right)^j \|\vec{\mu}^{(0)}\|$$

$$\leq \underbrace{\left(1 - \frac{\gamma t}{j}\right)^j}\|\vec{\mu}^{(0)}\|$$

$$\to e^{-\gamma t} \text{ as } j \to \infty$$