ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 1-3: L^∞-Norms *Are* Bounded by H^1-Seminorms in 1D

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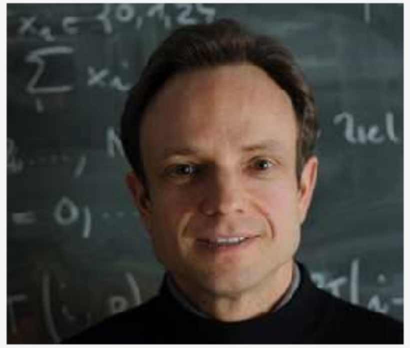

L^∞-norm: $\quad \|v\|_\infty := \max_{x\in\Omega} |v(x)|, \quad v \in C^0(\bar{\Omega})$

H^1-seminorm: $\quad |v|_1^2 := \int_\Omega \|\mathrm{grad}\, v\|^2 \, dx$

To show: $\quad \|v\|_\infty \leq |v|_1 \quad$ for $v \in C_0^1([0,1])$

NumPDE@ETHZ

Theorem 1.3.4.17. **First Poincaré-Friedrichs inequality**

If $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, *is* bounded, *then*

$$\|u\|_0 \leq \mathrm{diam}(\Omega)\, \|\mathbf{grad}\, u\|_0 \quad \forall u \in H_0^1(\Omega).$$

1D proof based on fundamental theorem of calculus

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

FTC $\quad \forall u \in C_0^\infty([0,1])$: $\quad u(x) = \underbrace{\cancel{u(0)}}_{=0} + \int_0^x \frac{du}{dx}(\tau)\,d\tau, \quad 0 \leq x \leq 1.$

$$\|u\|_0^2 = \int_0^1 \underbrace{\left|\int_0^x 1 \cdot \frac{du}{dx}(\tau)\,d\tau\right|^2}_{= u(x)^2} dx \overset{(1.3.4.15)}{\leq} \int_0^1 \left(\int_0^x 1\,d\tau \cdot \int_0^x \left|\frac{du}{dx}(\tau)\right|^2 d\tau\right) dx \leq \left\|\frac{du}{dx}\right\|_0^2.$$

Cauchy-Schwarz inequ.

FTC, because $u(0)=0$

$$|u(x)| = \left|\int_0^x 1 \cdot u'(\tau)\,d\tau\right| \leq \underbrace{\left(\int_0^x 1\,d\tau\right)^{\frac{1}{2}}}_{\leq 1} \left(\int_0^{\cancel{x}\,1} |u'(\tau)|^2\,d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\int_0^1 |u'(\tau)|^2\,d\tau\right)^{\frac{1}{2}} = |u|_{H^1(]0,1[)}.$$

$x \in [0,1]$ is arbitrary $\Rightarrow \forall x$

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Q: Where did we use $u(0) = \underbrace{u(1) = 0}$

$\hookrightarrow$ never used!

Warning: $d > 1$: $L^\infty \not\subset H^1$

point evaluation functional unbounded!

b)

Theorem 1.9.0.8. Multiplicative trace inequality (MTI)

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega).$$

1D: $\Omega =]a,b[\Rightarrow \|u\|^2_{L^2(\partial\Omega)} = u(a)^2 + u(b)^2$

Idea: Choose $\Omega =]0,x[$, $u \in C^1_{0,pw}(\Omega)$

$$u(x)^2 + \cancel{u(0)^2} \leq C\,\|u\|_{L^2(]0,x[)}\,\|u\|_{H^1(]0,x[)}$$

Now use that the norms increase when the interval increases

$$\leq C\,\|u\|_{L^2(]0,1[)}\,\|u\|_{H^1(]0,1[)}$$

$$\leq C\,\|u\|^2_{H^1(]0,1[)} \qquad \forall x \in]0,1[$$