ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 2-3: Pointwise "exact" Galerkin solution

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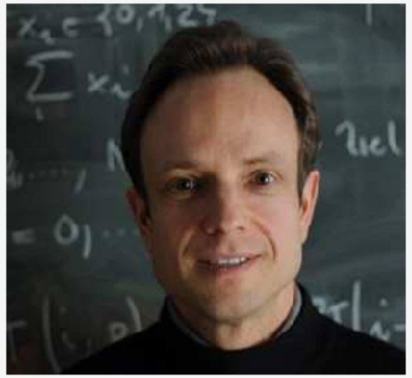

Based on Sect 2.2 & 2.3

$$u \in H_0^1([a,b]) : \int_a^b \frac{\partial u}{\partial x}(x)\,\frac{\partial v}{\partial x}(x)\,\mathrm{d}x = \int_a^b g(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1([a,b]) . \tag{2.3.1}$$

a) Strong form ?

Use i.b.p : Since $v(a) = v(b) = 0$

$$\Rightarrow \int_a^b \frac{d^2u}{d^2x}\cdot v\,dx = \int_a^b g\,v\,dx \quad \forall v \in C_0^\infty([a,b])$$

$\Rightarrow$ (Assume smooth u,g)

$$-\frac{d^2u}{d^2x} = g \quad \text{in }]a,b[$$
$$u(a) = u(b) = 0$$

b) Show : $u_h \in S_{1,0}^0(\mathcal{M})$ FE solution of (2.3.1)

$\mathcal{M} = \{a = x_0 < x_1 < \dots < x_M = b\}$,

$\Rightarrow \quad u_h(x_i) = u(x_i) \quad i = 0,\dots,M$

Idea: Show that

$$u_h = \sum_{i=1}^{N} u(x_i)\, b_h^i , \quad N := M-1$$

solves (2.3.1) for $v \in S_{1,0}^0(\mathcal{M})$. $\; b_h^i \leftarrow$ tent function

$$\int_a^b \frac{du_h}{dx}\,\frac{dv_h}{dx}\,dx = \int_a^b g\, v_h\, dx \quad \forall v_h \in \mathcal{S}^0_{1,0}(\mathcal{M})$$

$$\Leftrightarrow \underbrace{\int_a^b \frac{du_h}{dx}\cdot\frac{db_h^i}{dx}\,dx}_{(L)} = \underbrace{\int_a^b g\, b_h^i\, dx}_{(R)}, \quad i = 1,\dots,N$$

(L): Use
$$\frac{db_h^i}{dx} = \begin{cases} \frac{1}{h_i}, & x_{i-1} \le x \le x_i, \quad h_i = x_i - x_{i-1} \\ -\frac{1}{h_{i+1}}, & x_i \le x \le x_{i+1} \\ 0, & \text{elsewhere} \end{cases}$$

$$(L) = \frac{u(x_i)-u(x_{i-1})}{h_i} - \frac{u(x_{i+1})-u(x_i)}{h_{i+1}}$$

[only integration over $]x_{i-1},x_i[\,\cup\,]x_i,x_{i+1}[$ is required]

(R): Use ODE! $\quad g = -\frac{d^2u}{dx^2}$

$$-\int_{x_{i-1}}^{x_{i+1}} \frac{d^2u}{dx^2}\, b_h^i\, dx \overset{\text{i.b.p}}{=} \int_{x_{i-1}}^{x_{i+1}} \frac{du}{dx}\cdot\frac{db_h^i}{dx}\,dx = \int_{x_{i-1}}^{x_i} \frac{du}{dx}\,\frac{1}{h_i}\,dx - \int_{x_i}^{x_{i+1}} \frac{du}{dx}\,\frac{1}{h_{i+1}}\,dx$$

(L) = (R) by fundamental thm. of calculus

C, $\displaystyle\int_a^b g(x)\, v_h(x)\, dx$ one-sided limits

$$= \frac{1}{2} h \sum_{i=1}^{N} \big(g(x_i - 0) + g(x_i + 0)\big)\, v_h(x_i)$$

[composite trapezoidal rule on equidistant mesh]

For the argument in b) we need exact evaluation of the integrals defining the r.h.s. linear form.

CTR integrates $\mathcal{M}$-p.w. linear functions exactly

▷ No quadrature error, if g is $\mathcal{M}$-p.w. const

$\Leftrightarrow$ $g(x)\, v_h(x)$ is $\mathcal{M}$-p.w. linear

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