ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 8-3: Conservative finite-volume discretization based on Godunov numerical flux

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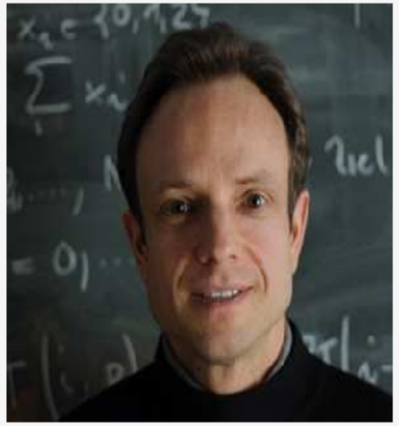


Cauchy problem :

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\sin(\pi u) = 0 \quad \text{on} \quad \mathbb{R}\times[0,T]\,, \tag{8.3.1}$$

with initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{R}$, satisfying

$$\underbrace{0 \leq u_0(x) \leq 1,}\quad x \in \mathbb{R}. \tag{8.3.2}$$

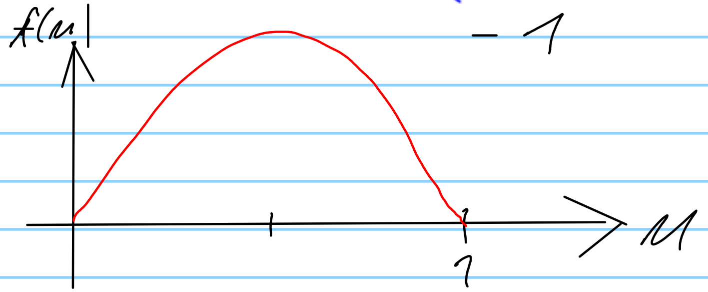
↳ By comparison thm: $u(x,t) \in [0,1]$ $\forall (x,t)$

▷ flux function : $f(u) = \sin(\pi u)$

a) Determine the entropy solutions of the Riemann problem with

(i) $u_0(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0, \end{cases}$, (ii) $u_0(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 0 & \text{for } x > 0, \end{cases}$

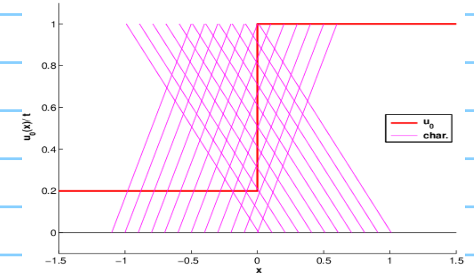
Note:



$f|_{[0,1]}$ concave $\Leftrightarrow$ $f'|_{[0,1]}$ decreasing

▷ Same situation as for the traffic flow model, $f(u) = u(1-u)$

(i)



$u_l < u_r$: intersecting characteristics

◁ intersecting characteristic

entropy shock solution

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Lemma 8.2.5.4. Shock solution of Riemann problem

For any two states $u_l, u_l \in \mathbb{R}$ the piecewise constant function

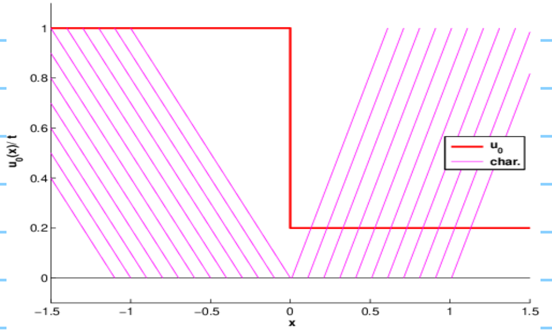
$$u(x,t) := \begin{cases} u_l & \text{for } x < \dot{s}t\,, \\ u_r & \text{for } x > \dot{s}t\,, \end{cases} \quad \dot{s} := \frac{f(u_l) - f(u_r)}{u_l - u_r}\,, \quad x \in \mathbb{R},\ 0 < t < T\,,$$

is a weak solution (→ Def. 8.2.3.4) of the related Riemann problem (→ Section 8.2.5) for the 1D scalar conservation law (8.2.2.1).

Shock speed $\dot{s} = f(1) - f(0) = 0$

$\rightarrow$ stationary shock

(ii)



$u_l > u_r$: diverging characteristics

◁ Rarefaction wave
entropy solution

Lemma 8.2.5.10. Rarefaction solution of Riemann problem

If $f \in C^2(\mathbb{R})$ is strictly $\begin{cases} \text{convex and} & u_l < u_r, \\ \text{concave and} & u_r < u_l, \end{cases}$ *then*

$$u(x,t) := \begin{cases} u_l & \text{for } x < \min\{f'(u_l), f'(u_r)\} \cdot t\,, \\ g(\frac{x}{t}) & \text{for } \min\{f'(u_l), f'(u_r)\} < \frac{x}{t} < \max\{f'(u_l), f'(u_r)\}\,, \\ u_r & \text{for } x > \max\{f'(u_l), f'(u_r)\} \cdot t\,, \end{cases}$$

$g := (f')^{-1}$, is a continuous weak solution of the Riemann problem (→ Def. 8.2.5).

$f'(u) = \pi \cos(\pi u)$

$f'(u_l) = -\pi\,, \quad f'(u_r) = \pi$

$g(\xi) = (f')^{-1}(\xi) = \frac{1}{\pi}\arccos(\frac{1}{\pi}\xi)$

$$\triangleright \quad u(x,t) = \begin{cases} 1 & \text{for } x < -\pi t\,, \\ \frac{1}{\pi}\arccos\left(\frac{x}{\pi t}\right) & \text{for } -\pi < x/t < \pi\,, \\ 0 & \text{for } x > \pi t\,. \end{cases}$$

b) Assume $\text{supp}(u_0) \subset [0,1]$ and (8.3.2). Describe the maximal support of the solution u of (8.3.1) in the x-t-plane.

- $f'(u)$ decreasing & $u \in [0,1]$

↳ speed of propagation $\dot{s}$

$\dot{s}_{min} = f'(1) = -\pi$

$\dot{s}_{max} = f'(0) = \pi$

$\text{supp}\, u(\cdot, t) \subset [0 - \pi t, 1 + \pi t]$

c, Godunov numerical flux $F_{GD}(v,w)$, $0 \le v,w \le 1$

Here: f concave

Intermediate state $\mu^{\downarrow}(v,w)$ from local Riemann problems

$v \le w$: shock

$$\mu^{\downarrow}(v,w) = \begin{cases} v, & \dot{s} \ge 0 \\ w, & \dot{s} \le 0 \end{cases}$$

$$\dot{s} = \frac{f(w)-f(v)}{w-v}$$

$v \ge w$: rarefaction

$g := (f')^{-1}$: $\mu^{\downarrow}(v,w) = \begin{cases} v, & f'(v) > 0 \\ g(0) = \frac{1}{2}, & f'(v) < 0 < f'(w) \\ w, & f'(w) < 0 \end{cases}$

transonic rarefaction

$$u^{\downarrow}(v,w) = \begin{cases} v & \text{, if } v < w \text{ and } \dot{s} > 0, \\ v & \text{, if } v \ge w \text{ and } \cos \pi v > 0, \\ w & \text{, if } v < w \text{ and } \dot{s} < 0, \\ w & \text{, if } v \ge w \text{ and } \cos \pi w < 0, \\ \frac{1}{2} & \text{, if } v \ge w \text{ and } \cos \pi v \le 0 \le \cos \pi w\,, \end{cases} \qquad \dot{s} := \frac{\sin(\pi w) - \sin(\pi v)}{w - v}.$$

$$F_{GD}(v,w) = f(\mu^{\downarrow}(v,w))$$

2-stage explicit RK-SSM:
$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

e, FV-MOL-ODE

$$(\mathcal{L}_h(\vec{\mu}(t)))_j = \frac{1}{h}\left(F(\mu_j(t),\mu_{j+1}(t)) - F(\mu_{j-1}(t),\mu_j(t))\right)$$
$j \in \mathbb{Z}$

$$\boxed{\vec{\kappa} = \mu^{(k)} + \tau \mathcal{L}_h(\vec{\mu}^{(k)}) \quad , \quad \vec{\mu}^{(k+1)} = \mu^{(k)} + \frac{\tau}{2}\left(\mathcal{L}_h(\mu^{(k)}) + \mathcal{L}_h(\vec{\kappa})\right).}$$

$$\begin{aligned} \kappa_j := (\vec{\kappa})_j &= \mu_j - \frac{\tau}{h}\left(F(\mu_j,\mu_{j+1}) - F(\mu_{j-1},\mu_j)\right), \\ (\mathcal{H}_h(\vec{\mu}))_j &= \mu_j - \frac{\tau}{2h}\left(F(\kappa_j,\kappa_{j+1}) - F(\kappa_{j-1},\kappa_j) + F(\mu_j,\mu_{j+1}) - F(\mu_{j-1},\mu_j)\right). \end{aligned} \tag{8.3.6}$$

2-point Godunov numerical flux

f/g, Relying on `sineClawRhs()` and `explTrpzTimestepping()` write a C++ function (in the file `finitevolumesineconslaw.h`)

```
template <typename RHSFUNCTOR>
Eigen::VectorXd solveSineConsLaw(
    RHSFUNCTOR &&g, unsigned int N,
    unsigned int M);
```

that solves the Cauchy problem for (8.3.1) numerically over the time interval $[0,1]$ with initial data

$$u_0(x) = \begin{cases} 1 & \text{for } 0 \le x < 1\,, \\ 0 & \text{elsewhere,} \end{cases}$$ ▷ $\operatorname{supp} u_0 \subset [0,1]$

using an equidistant spatial mesh with N cells and M uniform timesteps. Choose the spatial interval barely large enough such that the spatial support of $u(x,t)$ will be contained in it for all $0 \le t \le 1$. Sample the initial data u_0 at midpoints of mesh cells.

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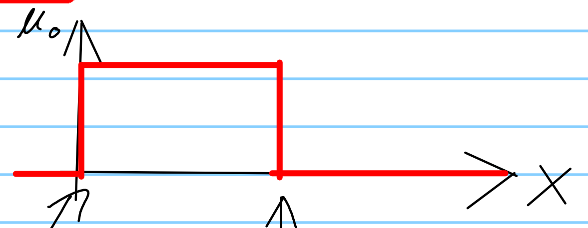
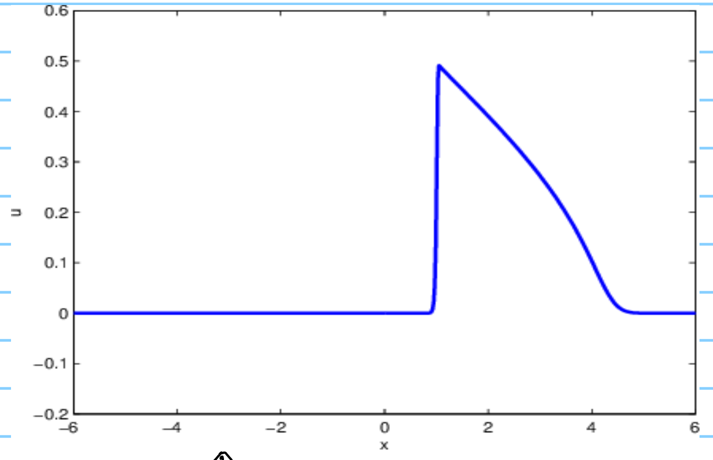
▷ $\operatorname{supp} u(\cdot,t) \subset [-\pi, 1+\pi]$ $\forall\, 0 \le t \le 1$

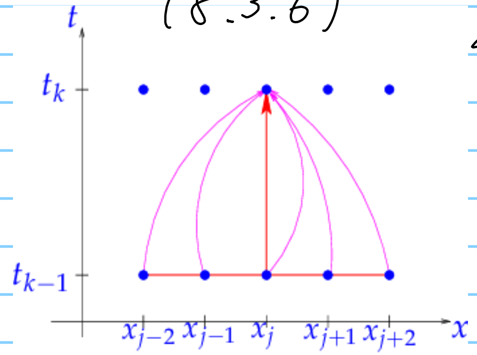
$\vec{\mu}(0) = [\mu_0(x_j)]_j$





Shock has started moving after interaction with left edge of rarefaction fan.

h) CFL timestep constraint (8.3.6)



2-point numerical flux & explicit trapezoidal rule

◁ Stencil of width $m = 2$

Formula from course

$\tau \le m \, h/\dot{s}$

$\dot{s} \mathrel{\hat{=}}$ maximal speed

▷ $\tau \le 2\, h/\pi$

i) Realize a C++ function (in the file `finitevolumesineconslaw.cc`)

```
unsigned int findTimestep();
```

that determines the maximum possible timestep size for the numerical experiment of Sub-problem (8-3.f). It returns the minimal number M of timesteps that barely avoids blow-up.

▷ Bisection based on blow-up detection

j) In the file `finitevolumesineconslaw.cc` create a modification

```
Eigen::VectorXd sineClawReactionRhs(
    const Eigen::VectorXd &mu, double c)
```

of the function `sineClawRhs()` from Sub-problem (8-3.d) so it still serves the same purpose but now for the augmented conservation law with reaction term

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\sin(\pi u) = -cu\,, \quad c \ge 0\,. \tag{8.3.12}$$

reaction term

▷ Add $-c\mu_j$ to the j-th component of the r.h.s. of FV-MOL-ODE

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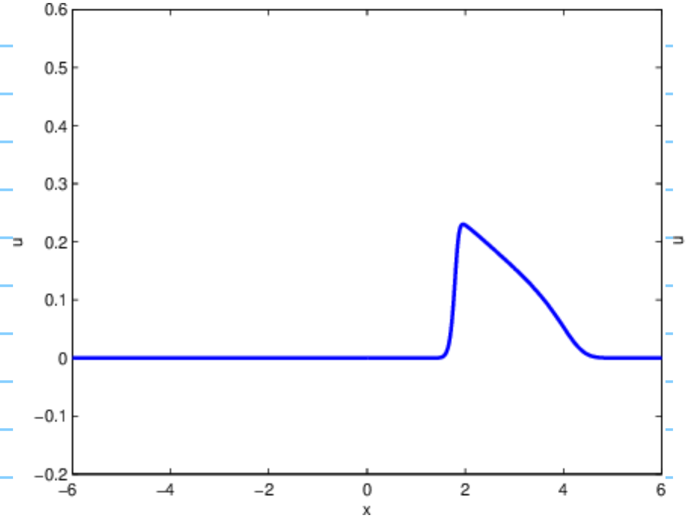
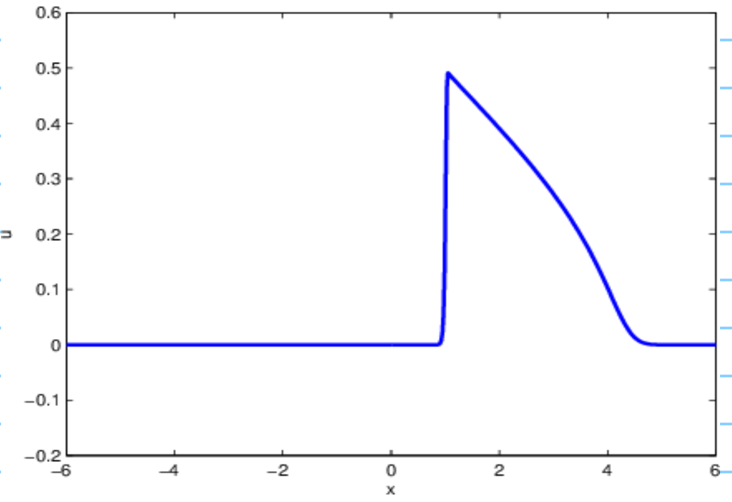
K,

$c = 1$



$c = 0$



Reaction term leads to a decay of the solution

$$\partial_t u = -cu$$

$$\Rightarrow \quad u(x,t) = e^{-ct}u_0(x)$$