ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 2-11: Hybrid-mesh Galerkin matrices and right-hand-side vectors

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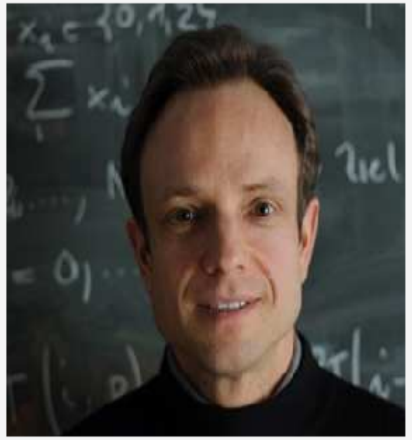
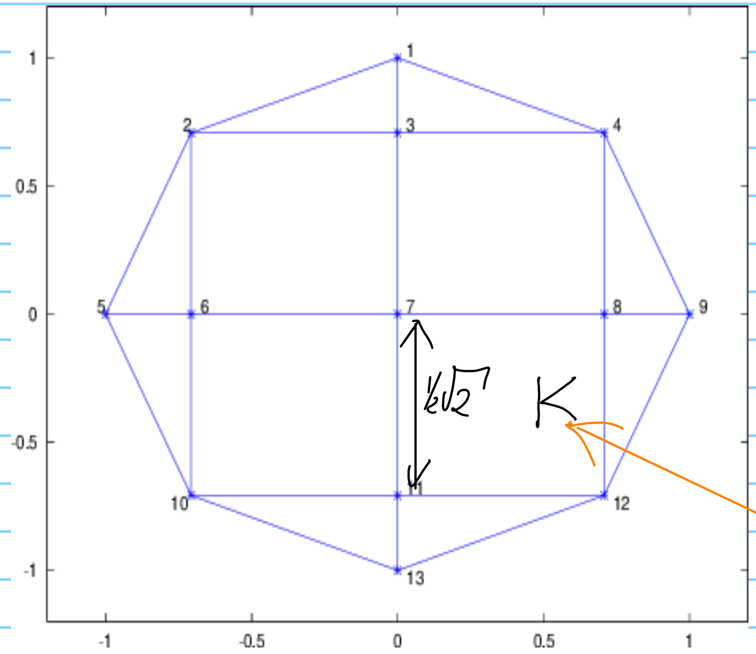


Useful to know: Sect. 2.4 & 2.6

• $S_0^1(\mathcal{M})$-Galerkin FEM for

$u \in H^1(\Omega)$ $\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \, dx = \int_\Omega f v \, dx \quad \forall v \in H^1(\Omega)$



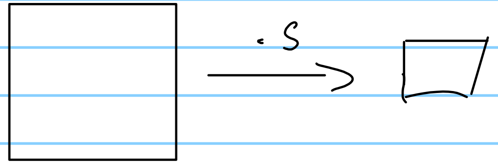
◁ $\mathcal{M}$

a) $\dim S_0^1(\mathcal{M}) = \#\mathcal{V}(\mathcal{M})$
$= 13$

← numbering of GSF of $S_1^0(\mathcal{M})$.

b) Element matrix A_K for square K

(i) Scaling argument



: $\int_K \operatorname{grad} b_K^j \cdot \operatorname{grad} b_K^i \, dx$

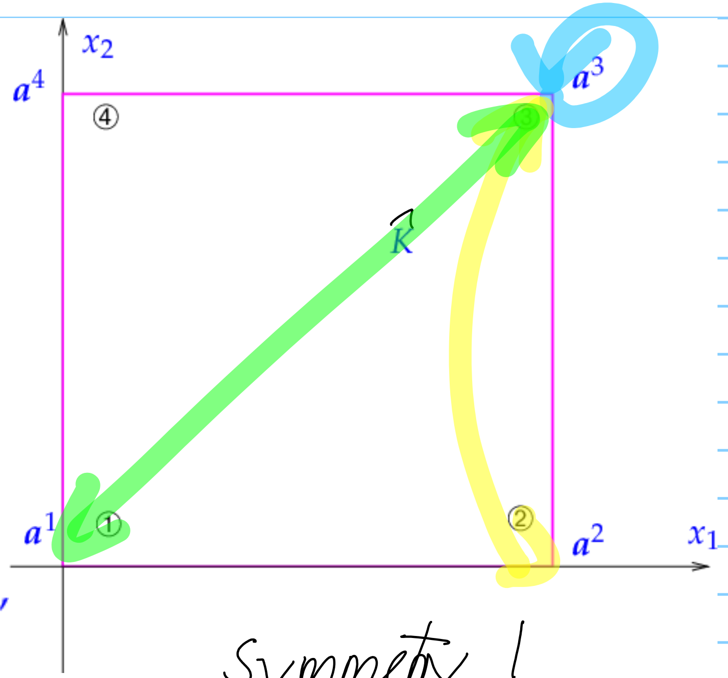
area $\cdot s^2$ ↗ K $\cdot 1/s$ $\cdot 1/s$

A_K is invariant w.r.t to all similarity trf.

②

(i) unit square:

$$b^1_K(\boldsymbol{x}) = (1-\hat{x}_1)(1-\hat{x}_2)\,,$$
$$b^2_K(\boldsymbol{x}) = \hat{x}_1(1-\hat{x}_2)\,,$$
$$b^3_K(\boldsymbol{x}) = \hat{x}_1\hat{x}_2\,,\qquad (2.6.2.3)$$
$$b^4_K(\boldsymbol{x}) = (1-\hat{x}_1)\hat{x}_2\,.$$



▶ $b^i_K(\boldsymbol{a}^j) = \delta_{ij}\,,\quad 1 \le i,j \le 4\,,$

$$\mathbf{grad}\,\hat{b}^1(\hat{x}_1,\hat{x}_2) = [\hat{x}_2-1,\hat{x}_1-1]^\top\,,$$
$$\mathbf{grad}\,\hat{b}^2(\hat{x}_1,\hat{x}_2) = [1-\hat{x}_2,-\hat{x}_1]^\top\,,$$
$$\mathbf{grad}\,\hat{b}^3(\hat{x}_1,\hat{x}_2) = [\hat{x}_2,\hat{x}_1]^\top\,,$$
$$\mathbf{grad}\,\hat{b}^4(\hat{x}_1,\hat{x}_2) = [-\hat{x}_2,1-\hat{x}_1]^\top\,.$$

Symmetry!

Sufficient to compute $(A_K)_{ij}$ for

$i=3,\ j=3 \rightarrow \int_{\hat{K}} \hat{x}_1^2+\hat{x}_2^2\,d\hat{x} = 2/3$

$i=3,\ j=2 \rightarrow \qquad = -1/6$

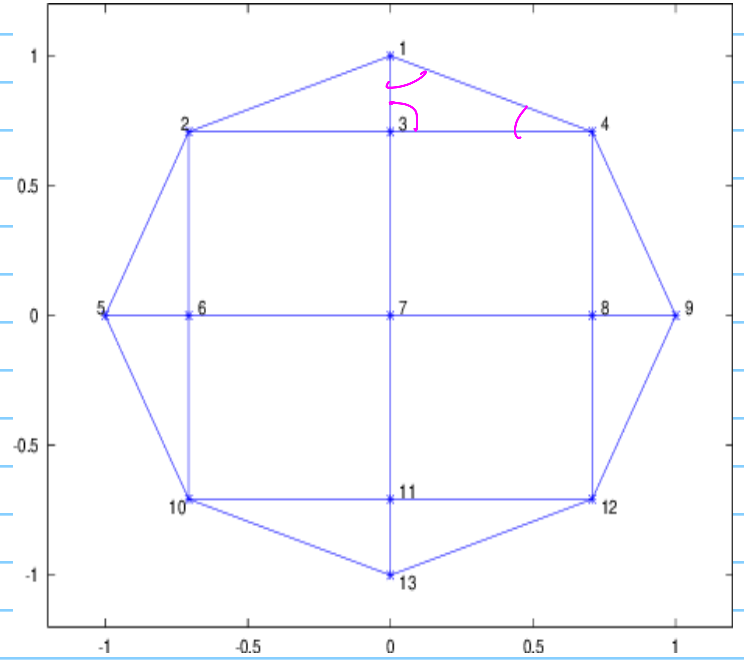
$i=3,\ j=1 \rightarrow \qquad = -1/3$

$$\mathbf{A}_{\hat{K}} = \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}. \qquad \forall \text{ squares } K! \qquad (2.11.3)$$

c) A_K for triangle K

$$\mathbf{A}_K = \frac{1}{2}\begin{bmatrix} \cot\omega_3+\cot\omega_2 & -\cot\omega_3 & -\cot\omega_2 \\ -\cot\omega_3 & \cot\omega_3+\cot\omega_1 & -\cot\omega_1 \\ -\cot\omega_2 & -\cot\omega_1 & \cot\omega_2+\cot\omega_1 \end{bmatrix}. \qquad (2.4.5.8)$$

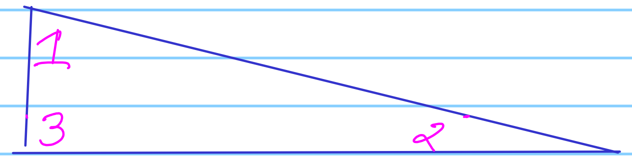
▷ Also invariant w.r.t. similarity tf.



$\cot\omega_1 = \sqrt{2}-1,$

$\cot\omega_2 = \dfrac{1}{\sqrt{2}-1},$

$\cot\omega_3 = 0,$



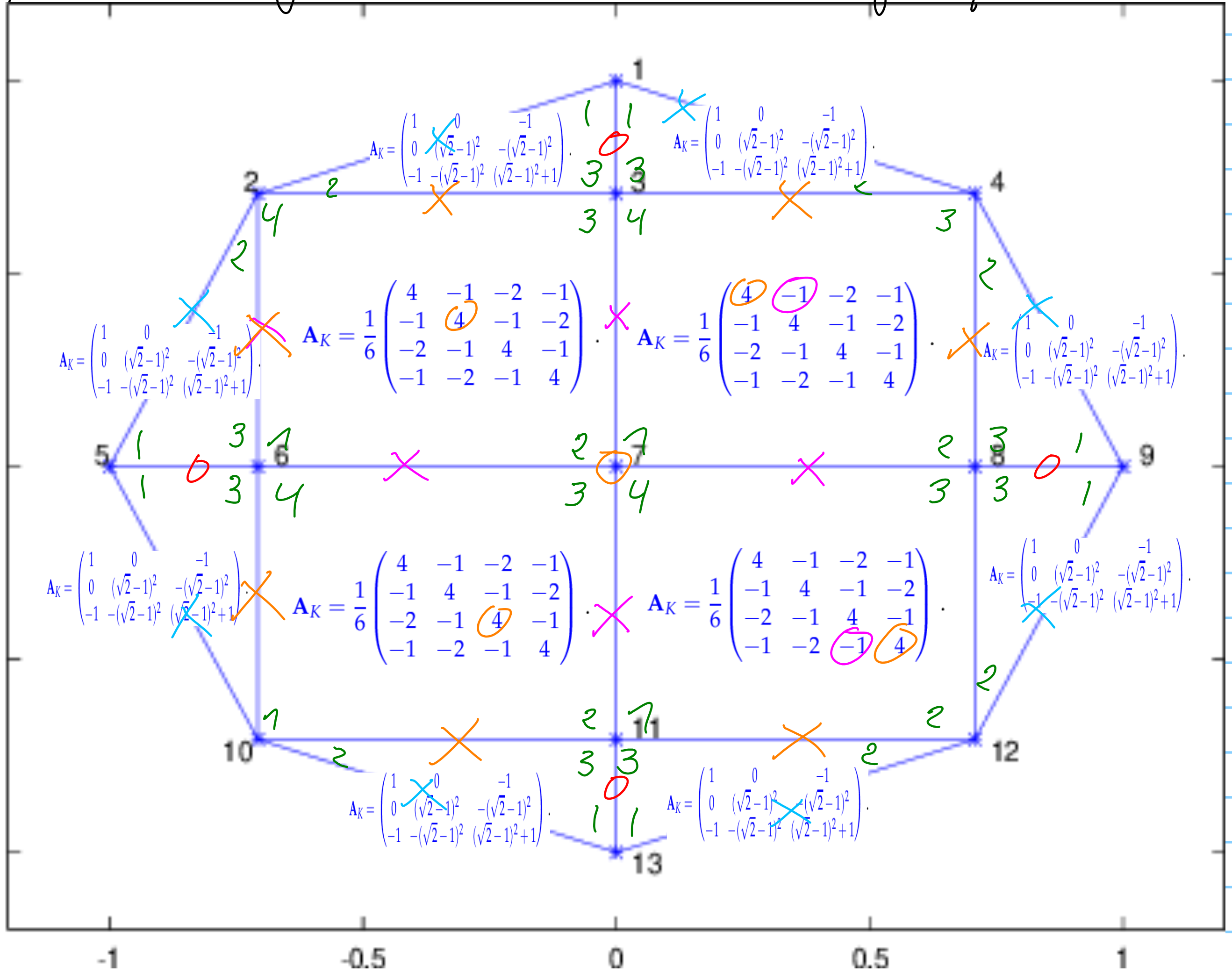
$$\mathbf{A}_K = \frac{1}{2}\begin{pmatrix} \cot\omega_3+\cot\omega_2 & -\cot\omega_3 & -\cot\omega_2 \\ -\cot\omega_3 & \cot\omega_3+\cot\omega_1 & -\cot\omega_1 \\ -\cot\omega_2 & -\cot\omega_1 & \cot\omega_2+\cot\omega_1 \end{pmatrix}$$
$$= \frac{1}{2(\sqrt{2}-1)}\begin{pmatrix} 1 & 0 & -1 \\ 0 & (\sqrt{2}-1)^2 & -(\sqrt{2}-1)^2 \\ -1 & -(\sqrt{2}-1)^2 & (\sqrt{2}-1)^2+1 \end{pmatrix}.$$

d) $\vec{\varphi}_K$ for square $K \in \mathcal{M}$ using

$$\int_K f(\mathbf{x})\,\mathrm{d}\mathbf{x} \approx \frac{|K|}{4}\sum_{i=1}^{4} f(\mathbf{a}_K^i), \quad |K| = 1/2 \qquad (2.11.4)$$

$$(\vec{\varphi}_K)_i = \tfrac{1}{8} f(a_K^i)$$

e) [Off-diagonal matrix entries $\leftrightarrow$ edges of $\mathcal{M}$]

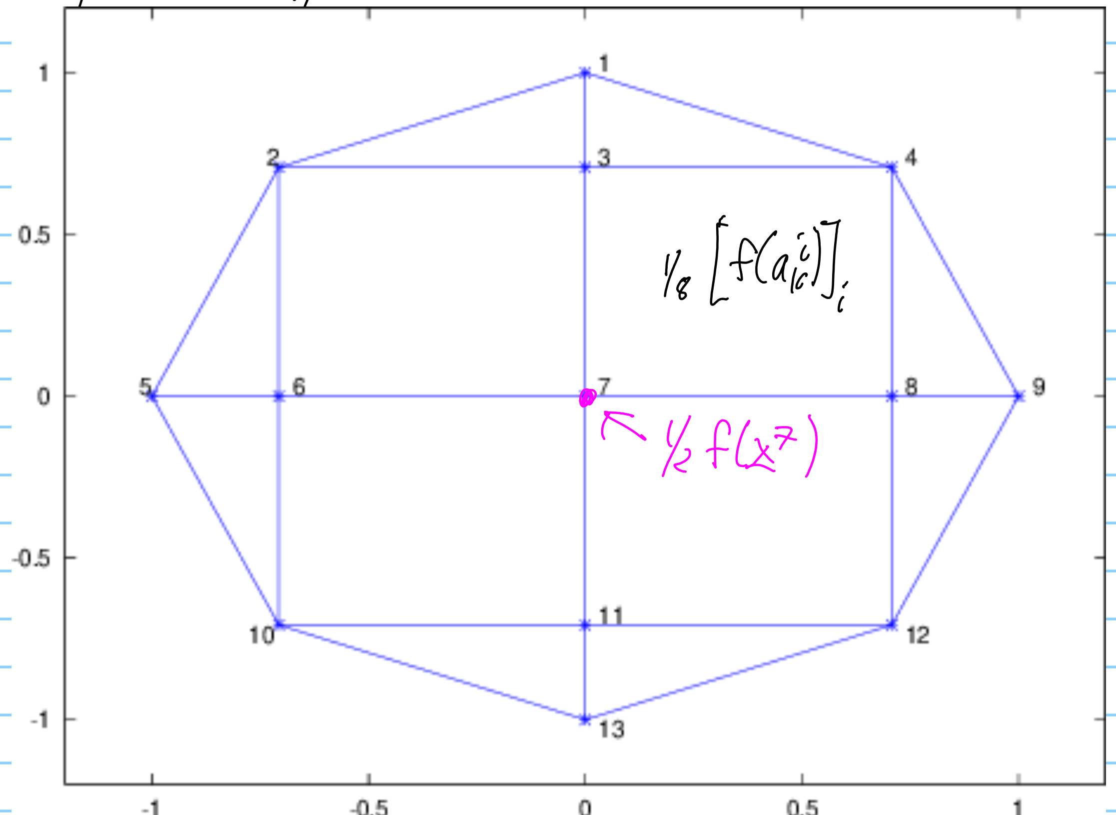


Use symmetry:

Define $p = 1/6$ and $q = \sqrt{2}-1$. Then,

$$\mathbf{A} = \begin{pmatrix}
\frac{1}{q} & 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4p+q & -p-\frac{q}{2} & 0 & 0 & -p-\frac{q}{2} & -2p & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{q} & -p-\frac{q}{2} & 8p+\frac{1}{q}(q^2+1) & -p-\frac{q}{2} & 0 & -2p & -2p & -2p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -p-\frac{q}{2} & 4p+q & 0 & 0 & -2p & -p-\frac{q}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{q} & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -p-\frac{q}{2} & -2p & 0 & -\frac{1}{q} & 8p+\frac{1}{q}(q^2+1) & -2p & 0 & 0 & -p-\frac{q}{2} & -2p & 0 & 0 \\
0 & -2p & -2p & -2p & 0 & -2p & 16p & -2p & 0 & -2p & -2p & -2p & 0 \\
0 & 0 & -2p & -p-\frac{q}{2} & 0 & 0 & -2p & 8p+\frac{1}{q}(q^2+1) & -\frac{1}{q} & 0 & -2p & -p-\frac{q}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{q} & \frac{1}{q} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -p-\frac{q}{2} & -2p & 0 & 0 & 4p+q & -p-\frac{q}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2p & -2p & -2p & 0 & -p-\frac{q}{2} & 8p+\frac{1}{q}(q^2+1) & -p-\frac{q}{2} & -\frac{1}{q} \\
0 & 0 & 0 & 0 & 0 & 0 & -2p & -p-\frac{q}{2} & 0 & 0 & -p-\frac{q}{2} & 4p+q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{q} & 0 & \frac{1}{q}
\end{pmatrix} \in \mathbb{R}^{13,13}$$

f) Compute $\vec{\varphi}$



Due to use of trapezoidal rule & cardinal basis property of GSFs

$\Downarrow$

$$(\vec{\varphi})_i = |K_i|\, f(x^i)$$

depending on adjacent cells.

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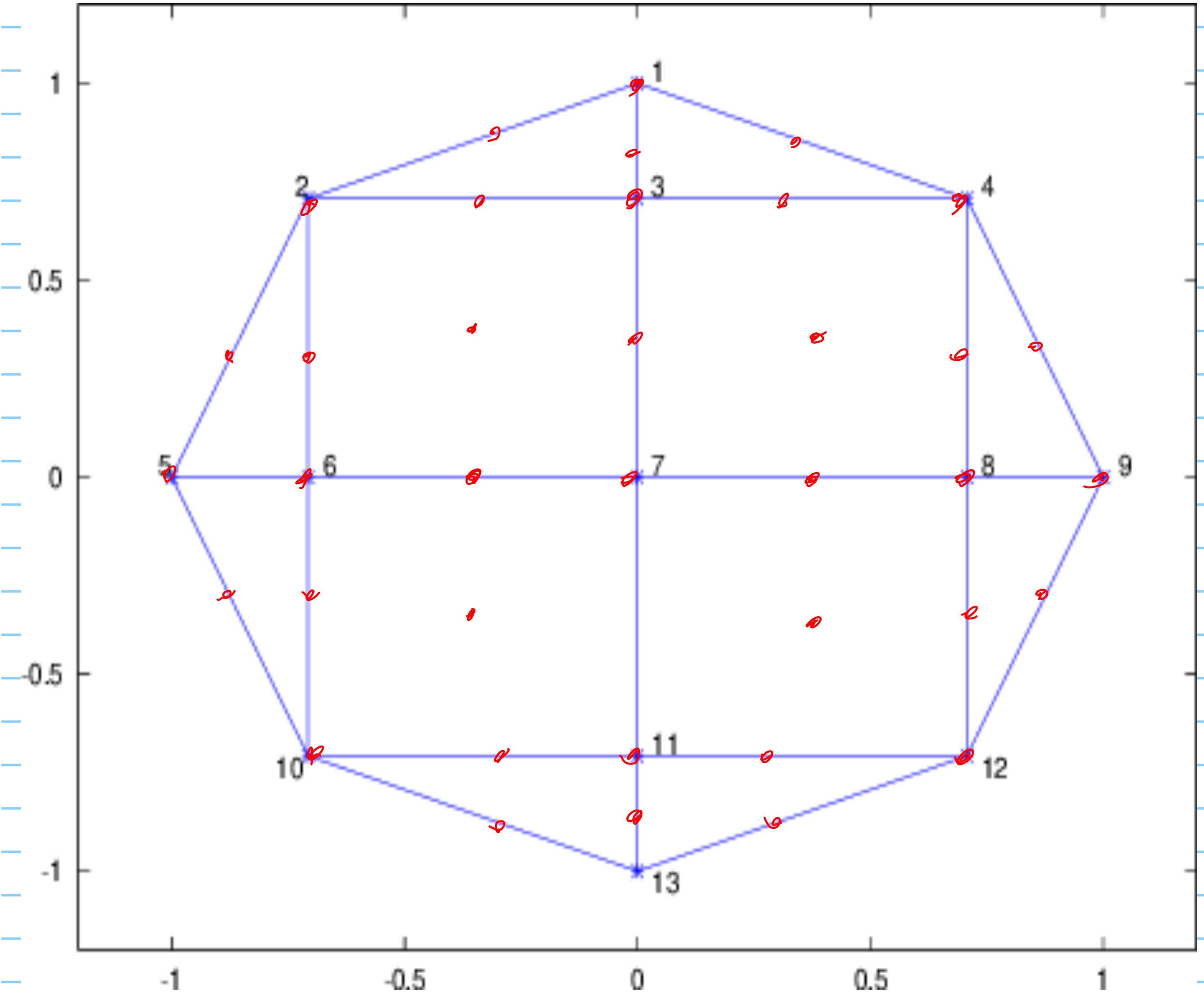
g, Now consider $S_2^0(\mathcal{M})$,

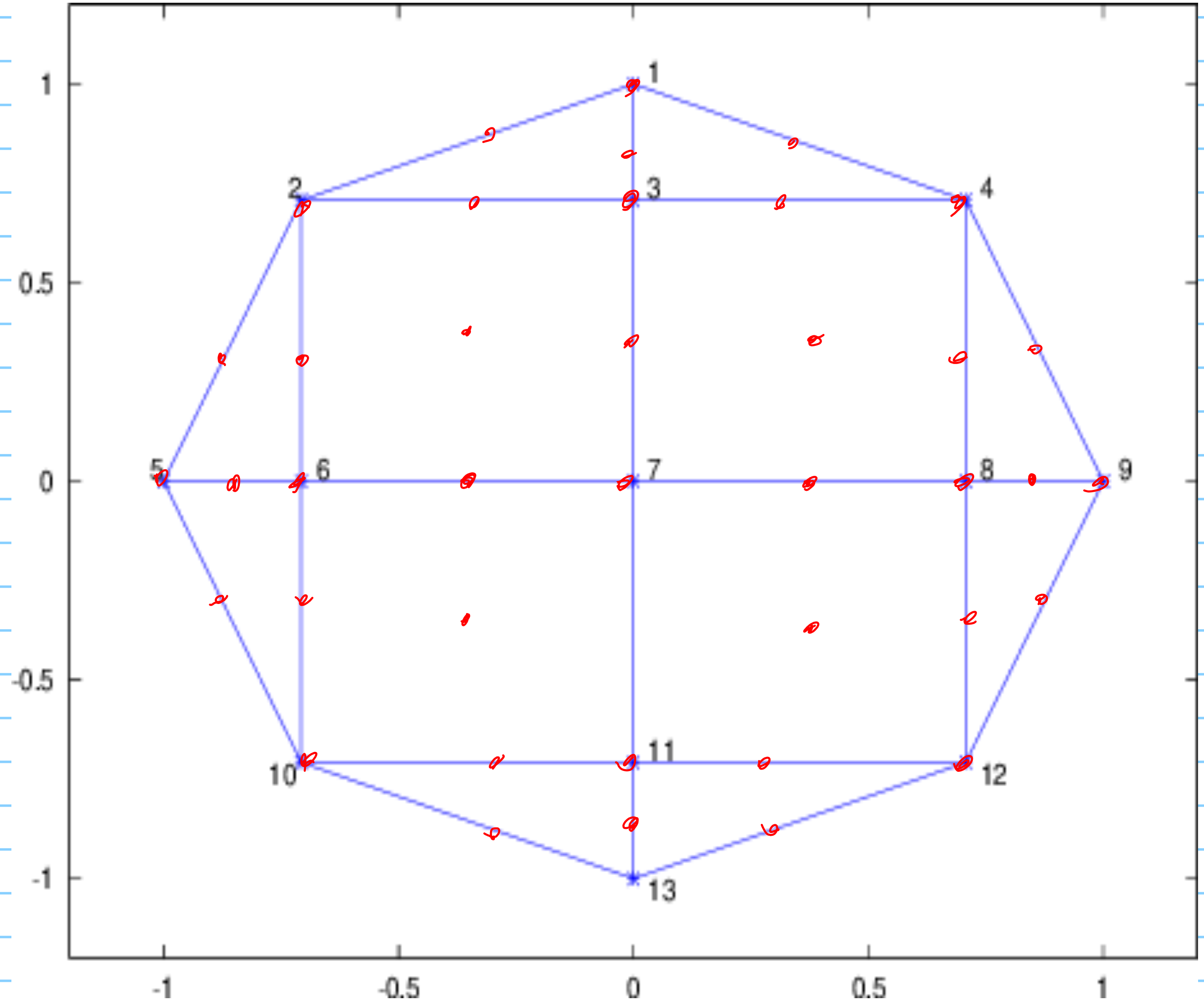
$\dim S_2^0(\mathcal{M}) = \#\mathcal{V}(\mathcal{M}) + \#\mathcal{E}(\mathcal{M}) + \#\{QUAD\} =$

$= 13 + 24 + 4 = 41$



h, $nnz(A) = 2$, where $A \exists$ $S_2^0(\mathcal{M})$-FE Gal. Mat.

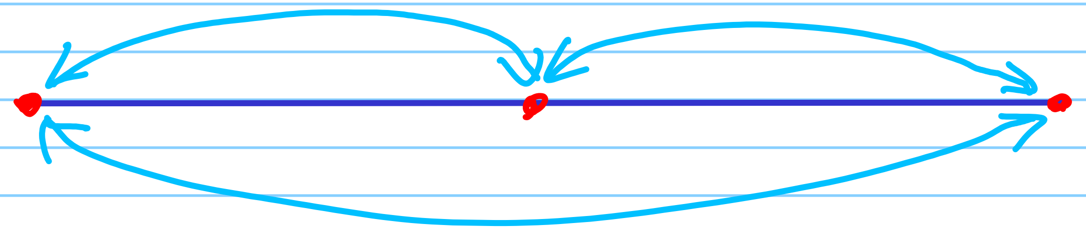
↳ A combinatorial bound



$\#\{\text{diagonal entries}\} = \#\dim S_2^0(\mathcal{M}) = 41$

Off-diagonal entries

- For every edge:



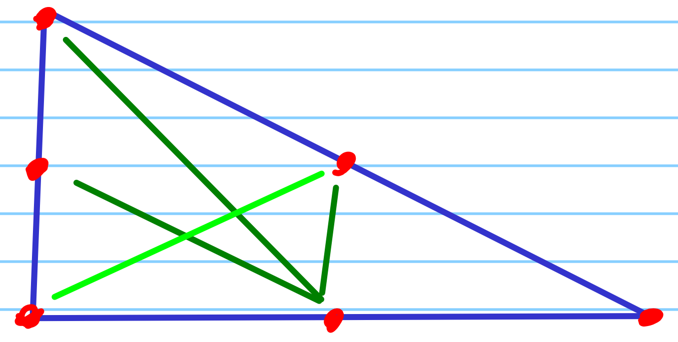
$2 \cdot 3$ n.z. offdiagonal entries

↳ symmetry of gal. mat.

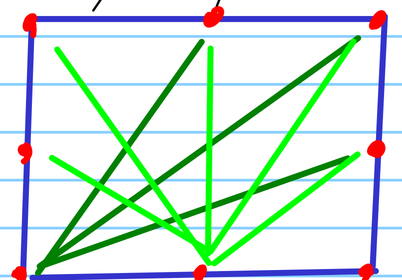
- For every triangle

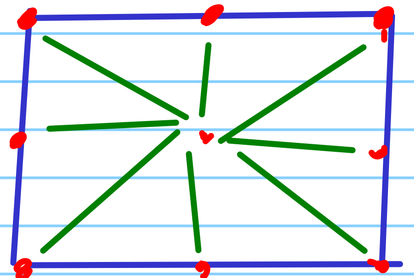
$2 \cdot (3 + 3 \cdot 3) = 24$

"interior interactions"



- For every square





$2 \cdot (4 \cdot 3 + 4 \cdot 5 + 8) = 80$

"interior interactions" for every square

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