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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 1-7: A second-order boundary value problem for vector fields

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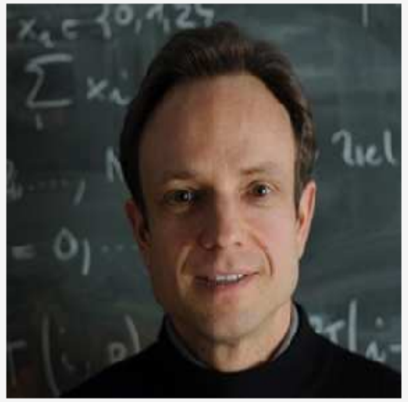

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$\Omega \subset \mathbb{R}^2$ bounded

classical function space

$$\underbrace{\int_\Omega \alpha(x)\,\operatorname{div}\mathbf{u}(x)\,\operatorname{div}\mathbf{v}(x) + \mathbf{u}(x)\cdot\mathbf{v}(x)\,\mathrm{d}x}_{=:\,a(u,v)} = \ell(\mathbf{v}) \quad \forall \mathbf{v} \in (C^1(\overline{\Omega}))^2 \qquad (1.7.1)$$

↳ uniformly positive definite: $\alpha(x) \geq \underline{\alpha} > 0 \quad \forall x \in \Omega$

a) Energy norm : [$a(\cdot,\cdot)$ s.p.d. is clear]

$$\|\mathbf{v}\|_a = \left(\int_\Omega \alpha(x)|\operatorname{div}\mathbf{v}(x)|^2 + \|\mathbf{v}(x)\|^2\,\mathrm{d}x\right)^{1/2}. \qquad (1.7.5)$$

b) "Energy space"

$$V_0 := \{v : \Omega \to \mathbb{R}^2 \text{ integrable} : \|\mathbf{v}\|_a < \infty\}.$$

following the idea of "maximal space for (1.7.1)"

c) Show that for $\mathbf{f} \in (L^2(\Omega))^2$ the concrete right-hand side functional

$$\ell(\mathbf{v}) := \int_\Omega \mathbf{f}(x)\cdot\mathbf{v}(x)\,\mathrm{d}x, \quad \mathbf{v} \in (C^1(\overline{\Omega}))^2, \qquad (1.7.6)$$

is continuous with respect to the energy norm $\|\cdot\|_a$ found in Sub-problem (1-7.a).

To show: $\exists C = C(f,\Omega) > 0 : |\ell(v)| \leq C\|v\|_a \quad \forall v \in V_0$

Tool: Cauchy-Schwarz inequ. in $(L^2(\Omega))^2$, applied componentwise

because: $\|v\|_0^2 = \|v_1\|_0^2 + \|v_2\|_0^2$

$$|\ell(v)| \underset{CSI}{\leq} \sum_{k=1}^{2} \left|\int_\Omega f_k(x)\,v_k(x)\,dx\right| \leq \sum_k \|f_k\|_0 \|v_k\|_0 \underset{CSI \text{ in } \mathbb{R}^2}{\leq} \underbrace{\|f\|_0}_{\leftrightarrow\, C} \|v\|_0$$

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d)

Another possibility for a right-hand side functional in (1.7.1) is

$$\ell(\mathbf{v}) := \int_{\partial\Omega} \mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\, \mathrm{d}S(\boldsymbol{x}) , \quad \mathbf{v} \in (C^1(\overline{\Omega}))^2 , \tag{1.7.7}$$

where $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^2$ is the exterior unit normal vector field.
Show that also this functional is continuous with respect to the energy norm $\|\cdot\|_a$ found in Subproblem (1-7.a).

Tools : • Gauss theorem

• CSI in $L^2(\partial\Omega)$

$$|\ell(\mathbf{v})| = \left|\int_{\partial\Omega} \mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\, \mathrm{d}S(\boldsymbol{x})\right| = \left|\int_{\Omega} 1 \,\operatorname{div} \mathbf{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right|$$

$$\overset{(*)}{\leq} \left(\int_\Omega 1^2 \,\mathrm{d}\boldsymbol{x}\right)^{1/2} \left(\int_\Omega |\operatorname{div} \mathbf{v}(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}\right)^{1/2} \leq |\Omega|^{1/2} \|\mathbf{v}\|_a .$$

e) Strong form of BVP induced by

$$\int_\Omega \alpha(\boldsymbol{x}) \operatorname{div} \mathbf{u}(\boldsymbol{x}) \operatorname{div} \mathbf{v}(\boldsymbol{x}) + \mathbf{u}(\boldsymbol{x}) \cdot \mathbf{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}\boldsymbol{x} \quad \forall \mathbf{v} \in (C^1(\overline{\Omega}))^2 \tag{1.7.1}$$

[assume sufficiently smooth $\underline{u}$, $\underline{f}$]

(I) Test with $\underline{v} \in (C_0^\infty(\overline{\Omega}))^2$

+ IBP ($\leftrightarrow$ Green's formula)

$$\int_\Omega -\mathbf{grad}\,(\alpha(\boldsymbol{x}) \operatorname{div} \mathbf{u}) \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\boldsymbol{x} + \underbrace{\cancel{\int_{\partial\Omega} \alpha(\boldsymbol{x}) \operatorname{div} \mathbf{u}(\boldsymbol{x})(\mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}))\, \mathrm{d}S(\boldsymbol{x})}}_{=0 \text{ since } v_{|\partial\Omega}} = \int_\Omega \mathbf{f}(\boldsymbol{x}) \cdot \mathbf{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} .$$

▷ PDE: $-\mathbf{grad}\, \alpha(x)\, \operatorname{div} \underline{u} + \underline{u} = f$ in Ω (*)

(II) Extract "natural" BDC

Test with $\underline{v} \in (C^\infty(\overline{\Omega}))^2$ & IBP

$$\cancel{\int_\Omega -\mathbf{grad}\,(\alpha(\boldsymbol{x}) \operatorname{div} \mathbf{u}) \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\boldsymbol{x}} + \int_{\partial\Omega} \alpha(\boldsymbol{x}) \operatorname{div} \mathbf{u}(\boldsymbol{x})(\mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}))\, \mathrm{d}S(\boldsymbol{x}) = \cancel{\int_\Omega \mathbf{f}(\boldsymbol{x}) \cdot \mathbf{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}} .$$

due to PDE (*)

▷ $$\int_{\partial\Omega} \alpha(x) \operatorname{div} \underline{u}(x) \underbrace{(\underline{v}(x) \cdot n(x))}_{\text{can be a rather general functions}} dS(x) = 0 \quad \forall \underline{v} \in (C^\infty(\overline{\Omega}))^2$$

▷ $\cancel{\alpha(x)} \operatorname{div} \underline{u}(x) = 0$ on $\partial\Omega$

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$$\int_\Omega \alpha(\boldsymbol{x})\,\operatorname{div}\mathbf{u}(\boldsymbol{x})\,\operatorname{div}\mathbf{v}(\boldsymbol{x}) + \mathbf{u}(\boldsymbol{x})\cdot\mathbf{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \ell(\mathbf{v}) := \int_{\partial\Omega}\mathbf{v}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) \quad \forall \mathbf{v}\in(C^1(\overline{\Omega}))^2$$

(I) PDE: Test with $v \in (C^1_0(\overline{\Omega}))^2$ & IBP

$$-\operatorname{div}\alpha(x)\operatorname{grad}u + u = 0$$

(II) From IBC with $v \in (C^\infty(\overline{\Omega}))^2$:

$$\int_\Omega \underbrace{-\mathbf{grad}(\alpha(\boldsymbol{x})\operatorname{div}\mathbf{u})\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}}_{=0}\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega}\alpha(\boldsymbol{x})\operatorname{div}\mathbf{u}(\boldsymbol{x})\,\mathbf{v}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) = \int_{\partial\Omega}\mathbf{v}(\boldsymbol{x})\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}).$$

$$\triangleright \quad \int_{\partial\Omega}\underbrace{(\alpha(\boldsymbol{x})\operatorname{div}\mathbf{u}(\boldsymbol{x}))}_{\in C^0(\partial\Omega) \text{ by smoothness assumption}}\mathbf{v}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) = \int_{\partial\Omega} 1\,\mathbf{v}(\boldsymbol{x})\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}), \quad \forall v$$

$$\boxed{\alpha(\boldsymbol{x})\operatorname{div}\mathbf{u}(\boldsymbol{x}) = 1, \quad \boldsymbol{x}\in\partial\Omega} . \qquad (1.7.12)$$

g) Quadratic minimization problem is associated with

$$\int_\Omega \alpha(\boldsymbol{x})\,\operatorname{div}\mathbf{u}(\boldsymbol{x})\,\operatorname{div}\mathbf{v}(\boldsymbol{x}) + \mathbf{u}(\boldsymbol{x})\cdot\mathbf{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \ell(\mathbf{v}) := \int_{\partial\Omega}\mathbf{v}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) \quad \forall \mathbf{v}\in(C^1(\overline{\Omega}))^2$$

Abstract for s.p.d. $a(\cdot,\cdot)$

$$u \in V_0 : a(u,v) = \ell(v) \iff u = \operatorname*{argmin}_{v\in V_0} \tfrac{1}{2}a(v,v) - \ell(v)$$

$$\boxed{\mathbf{u} = \operatorname*{argmin}_{\mathbf{v}\in(C^1(\overline{\Omega}))^2} \tfrac{1}{2}\int_\Omega |\operatorname{div}\mathbf{v}(\boldsymbol{x})|^2 + \|\mathbf{v}\|^2\,\mathrm{d}\boldsymbol{x} - \int_{\partial\Omega}\mathbf{v}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x})} . \qquad (1.7.15)$$

h) "Reverse engineering"

Find $\ell(\cdot)$ such that a prescribed u solves

$$\int_\Omega \cancel{\alpha(\boldsymbol{x})}^{=1}\,\operatorname{div}\mathbf{u}(\boldsymbol{x})\,\operatorname{div}\mathbf{v}(\boldsymbol{x}) + \mathbf{u}(\boldsymbol{x})\cdot\mathbf{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \ell(\mathbf{v}) \quad \forall \mathbf{v}\in(C^1(\overline{\Omega}))^2 \qquad (1.7.1)$$

Here:

$$\mathbf{u}(\boldsymbol{x}) := \begin{bmatrix}\sin(x_1)\\ \cos(x_2)\end{bmatrix}, \quad \boldsymbol{x} := \begin{bmatrix}x_1\\ x_2\end{bmatrix} \in \mathbb{R}^2 . \qquad (1.7.16)$$

$$\int_\Omega \operatorname{div}u\cdot\operatorname{div}v + u\,v\,dx \overset{\text{IBP}}{=}$$

$$= \int_\Omega (-\operatorname{grad}\operatorname{div}u + u)\,v\,dx + \int_{\partial\Omega}\operatorname{div}u\,(v\cdot n)\,dS$$

$\downarrow$ compute directly $\downarrow$

$$= \int_\Omega 2\begin{bmatrix}\sin(x_1)\\ \cos(x_2)\end{bmatrix}\cdot v(x)\,dx + \int_{\partial\Omega}(\cos(x_1)-\sin(x_2))(v\cdot n)\,dS$$

$$=: \ell(v)$$

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