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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial
Problem 1-2: Linear functionals on Sobolev spaces

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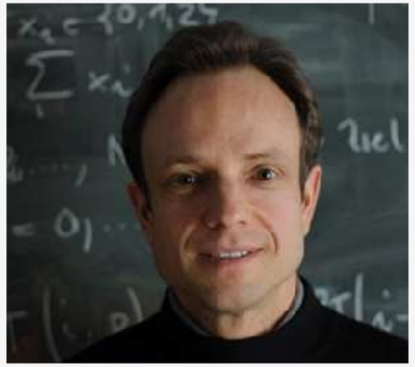

$$\ell_1(v) := \int_\Omega \mathbf{c} \cdot \mathbf{grad}\, v(x)\, dx\,, \quad \mathbf{c} \in \mathbb{R}^2\,, \tag{1.2.1}$$

$$\ell_2(v) := \int_\Omega v(x)\, dx\,, \tag{1.2.2}$$

$\Omega = \{\|x\| < 1\} \subset \mathbb{R}^2$

$$\ell_3(v) := \int_{\partial\Omega} \mathbf{grad}\, v(x) \cdot \boldsymbol{n}(x)\, dS(x)\,, \tag{1.2.3}$$

$$\ell_4(v) := \int_\Omega v\left(\frac{x}{\|x\|}\right) dx\,. \tag{1.2.4}$$

a) Continuity of ℓ on $L^2(\Omega)$

$\exists C > 0: \quad |\ell(v)| \leq C \|v\|_{L^2(\Omega)} \quad \forall v \in L^2(\Omega)$

In order to show that a functional is unbounded on $L^2(\Omega)$, one tries to find a function $v \in L^2(\Omega)$, which can be verified by showing $\|v\|_{L^2(\Omega)} < \infty$, for which $|\ell(v)| = \infty$, which often manifests itself as the divergence of an improper integral.

- ℓ_1 : Suspect not continuous, because of presence of derivatives

  $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v(x) = \log|x_1| \rightarrow \|v\|_0 < \infty$

  $\mathbf{grad}\, v(x) = \begin{bmatrix} 1/x \\ 0 \end{bmatrix} \rightarrow$ "$\ell_1(v) = \infty$"

- ℓ_2 : continuous, by Cauchy-Schwarz inequ.,

  $\int_\Omega f(x) g(x)\, dx \leq \|f\|_0 \|g\|_0$

  $\Rightarrow \quad |\ell_2(v)| = \int_\Omega 1 \cdot v\, dx \leq |\Omega|^{1/2} \|v\|_0$

- ℓ_3 : no continuity : $v(r) = \log(1-r) \rightarrow \|v\|_0 < \infty$, $r = \|x\|$

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$\operatorname{grad} v(r) = \underline{e}_r \frac{1}{1-r}$ "$= \infty$" for $\underbrace{r = 1}_{\leftrightarrow \partial\Omega}$

$\Rightarrow$ "$\ell_3(v) = \infty$"

- $\ell_4(v) = \int_\Omega v\left(\frac{x}{\|x\|}\right) dx$

polar cordinates

$$\ell_4(v) = \int_\Omega v\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)\mathrm{d}\boldsymbol{x} = \int_0^1\int_0^{2\pi} v\left(\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}\right) r\,\mathrm{d}\phi\mathrm{d}r =$$

$$= \tfrac{1}{2}\int_0^{2\pi} v\left(\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}\right)\mathrm{d}\phi = \tfrac{1}{2}\int_{\partial\Omega} v(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}).$$

$v(x) = \log(1 - \|x\|)$ $\rightarrow$ $\|v\|_0 < \infty$; $\rightarrow$ "$\ell_4(v) = \infty$"

b)

In order to show that a functional is unbounded on $\cancel{L^2(\Omega)}$ $H^1(\Omega)$, one tries to find a function $v \in \cancel{L^2(\Omega)}$ $H^1(\Omega)$, which can be verified by showing $\|v\|_{H^1(\Omega)} < \infty$, for which $|\ell(v)| = \infty$, which often manifests itself as the divergence of an improper integral.

Remark: $\|v\|_0 \leq \|v\|_1 \quad \forall v \in H^1(\Omega)$

$\Rightarrow$ Better chances for continuity on H^1

$$\ell_1(v) := \int_\Omega \mathbf{c}\cdot\mathbf{grad}\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \quad \mathbf{c}\in\mathbb{R}^2\,, \tag{1.2.1}$$

$$\ell_2(v) := \int_\Omega v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \tag{1.2.2}$$

$$\ell_3(v) := \int_{\partial\Omega} \mathbf{grad}\, v(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x})\,, \tag{1.2.3}$$

$$\ell_4(v) := \int_\Omega v\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)\mathrm{d}\boldsymbol{x}\,. \tag{1.2.4}$$

- ℓ_1 : continuity by CSI

$\int_\Omega c\cdot\operatorname{grad} v(x)\,dx \leq \int_\Omega \|c\|\,\|\operatorname{grad} v(x)\|\,dx$

$\leq \|c\|\,|\Omega|^{1/2}\,\|v\|_1$

- ℓ_2 : continuity inherited from L^2
- ℓ_3 : Heuristics: $v \in H^1(\Omega) \Rightarrow \operatorname{grad} v \in L^2(\Omega)$

no bounded restriction to $\partial\Omega$

$v(r) = (1-r)\log(1-r)$

$\operatorname{grad} v(r) = \underline{e}_r\,(1 - \log(1-r)) \rightarrow \|v\|_1 < \infty$

(3) "$\mathbf{grad}\, v(x) \cdot e_r = \infty$" for $\|x\| = 1$

$\hookrightarrow = n \quad \Rightarrow \quad \ell_3(v) = \infty$

- $\ell_4(v) = \frac{1}{2} \int_{\partial\Omega} v(x)\, dS(x)$

Hint: Use a trace theorem for H^1

Theorem 1.9.0.8. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \underbrace{\|u\|^2_{L^2(\partial\Omega)} \leq C \|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)}} \quad \forall u \in H^1(\Omega)\,.$$

$$\Rightarrow \quad \|v_{|\partial\Omega}\|_{L^2(\Omega)} \leq C \|v\|_{H^1(\Omega)}$$

$$\ell_4(v) = \frac{1}{2} \int_{\partial\Omega} 1 \cdot v(x)\, dS(x) \underset{\substack{\uparrow \\ \text{CSI in } L^2(\partial\Omega)}}{\leq} \tfrac{1}{2} |\partial\Omega|^{1/2} \|v\|_{L^2(\partial\Omega)}$$

$$\leq C \|v\|_{H^1(\Omega)}$$

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