ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 3-7: Maximum principle for reaction-diffusion boundary value problems

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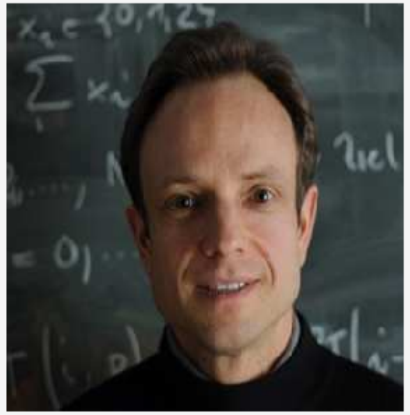


Dirichlet BVP :
$$-(1-c)\Delta u + cu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.7.1}$$

a, Variational formulation :

LVP: $u \in H_0^1(\Omega)$:
$$\underbrace{\int_\Omega (1-c)\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v + c\,u\,v \,\mathrm{d}x}_{=:a(u,v)} = \underbrace{\int_\Omega f\,v\,\mathrm{d}x}_{=:\ell(v)} \quad \forall v \in H_0^1(\Omega)\,. \tag{3.7.2}$$

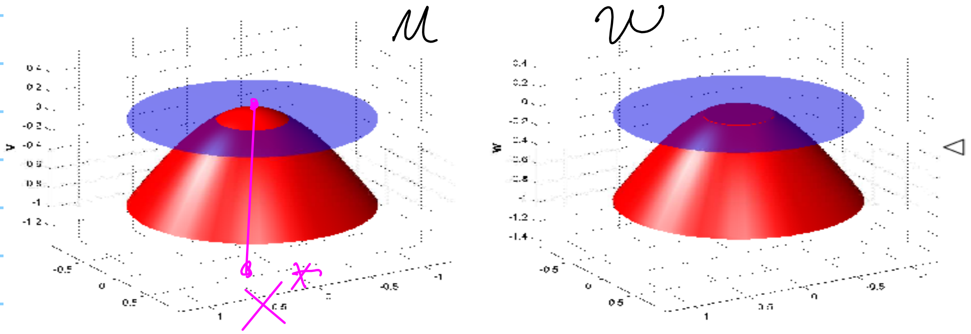
b, $\Leftrightarrow$ QMP for $J(v) = \frac{1}{2}a(v,v) - \ell(v)$

$$u = \underset{w \in H_0^1(\Omega)}{\operatorname{argmin}} \frac{1}{2}\int_\Omega (1-c)\|\mathbf{grad}\,w(x)\|^2 + c\,w(x)^2 \,\mathrm{d}x - \int_\Omega f(x)\,w(x)\,\mathrm{d}x\,.$$

c, Following the reasoning of [Lecture → Section 3.7.1] argue why

$$\max_{x\in\Omega} u(x) = 0 \tag{3.7.3}$$

holds. Why is $f \le 0$ important for your argument? [A maximum principle]



◁ A visual cue for the proof requested in Sub-problem (3-7.c).

$\triangleright$ $w \le u$

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Assume:

$u(x^*) = \max_{x\in\Omega} u(x) > 0$ for some $x^* \in \Omega$

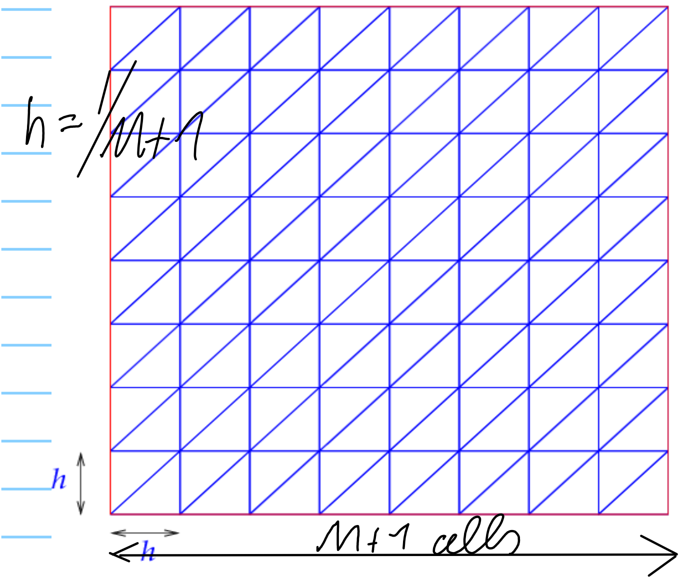
$w(x) = \min(u(x), 0)$

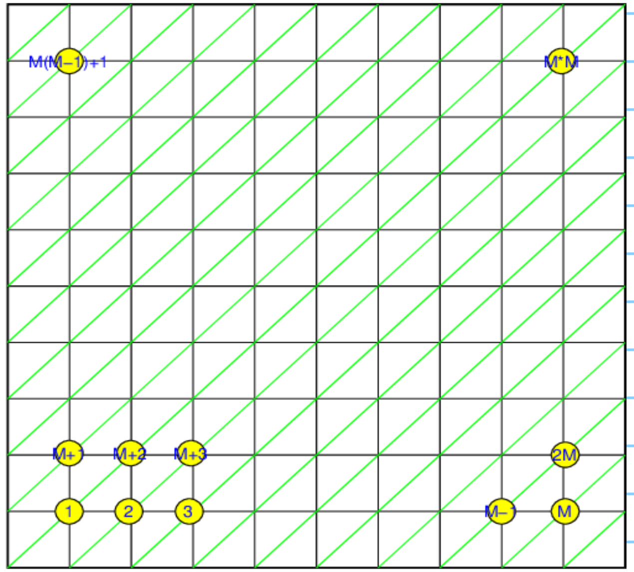
$$u = \operatorname*{argmin}_{v=0 \text{ on } \partial\Omega} \underbrace{\frac{1}{2}\int_\Omega (1-c)\|\mathbf{grad}\, v\|^2 + c|v|^2}_{\text{smaller for } w} + \underbrace{\overbrace{(-f)}^{\geq 0} v \,\mathrm{d}x}_{\leq \text{ for } w} \tag{3.7.4}$$

$J(w) < J(u)$

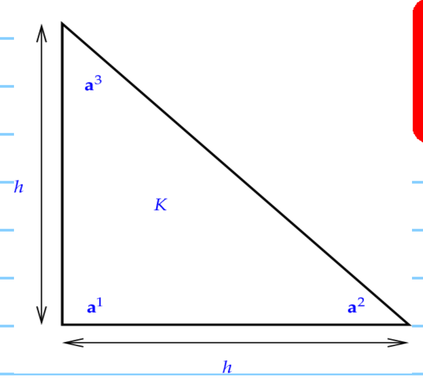
contradicts u being a minimizer.

$\mathcal{S}_1^0(\mathcal{M})$-FEM on tensor-product mesh





d) All triagles of $\mathcal{M}$ have the same shape and, therefore, will give rise to the same element matrices for the bilinear form $(u,v) \mapsto \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x$ and the lowesr-order Lagrangian finite element space $\mathcal{S}_1^0(\mathcal{M})$. Compute this element matrix.



$$\mathbf{grad}\,\lambda_1 = h^{-1}\sqrt{2}\begin{bmatrix}-1\\-1\end{bmatrix},$$

$$\mathbf{grad}\,\lambda_2 = h^{-1}\begin{bmatrix}1\\0\end{bmatrix},$$

$$\mathbf{grad}\,\lambda_3 = h^{-1}\begin{bmatrix}0\\1\end{bmatrix},$$

$$\mathbf{A}_K = \frac{1}{2}\begin{bmatrix}2 & -1 & -1\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}. \tag{3.7.5}$$

Check: row/column sums vanish

e) Compute the $\mathcal{S}_1^0(\mathcal{M})$-element matrix for the triangles of $\mathcal{M}$ and the bilinear form $(u,v) \mapsto \int_\Omega u\, v \,\mathrm{d}x$. This element matrix is also known as **element mass matrix**.

$$\mathbf{M}_K = \frac{1}{12}|K|\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}. \tag{3.7.6}$$

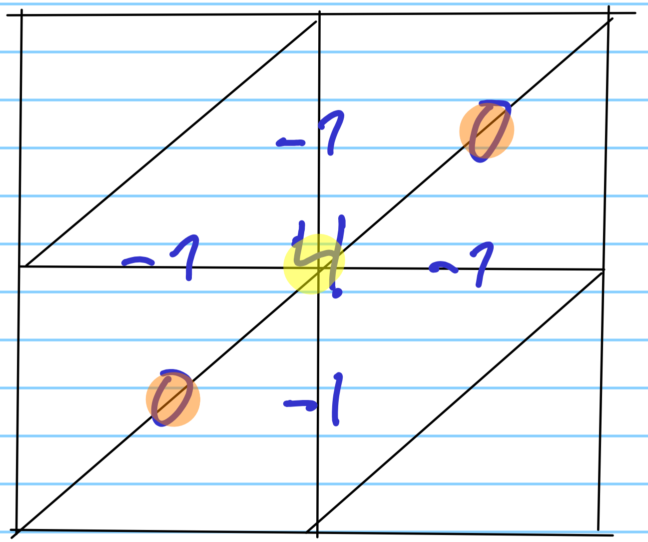
f) $\dim \mathcal{S}_{1,0}^{0}(\mathcal{M}) = \#\{\text{interior nodes}\} = M^2$

③ Write a C++ function

g,
```
Eigen::SparseMatrix<double>
    computeGalerkinMatrix(unsigned int M,double c);
```

that initializes the finite element Galerkin matrix for (3.7.1), the mesh from Fig. 30 and the finite element space $\mathcal{S}^0_{1,0}(\mathcal{M})$, assuming lexikographic numbering of global shape functions. The argument M passes the number of nodes in each direction, see Fig. 31, and c the constant $c \in]0,1[$.

For contribution of $-\Delta$:



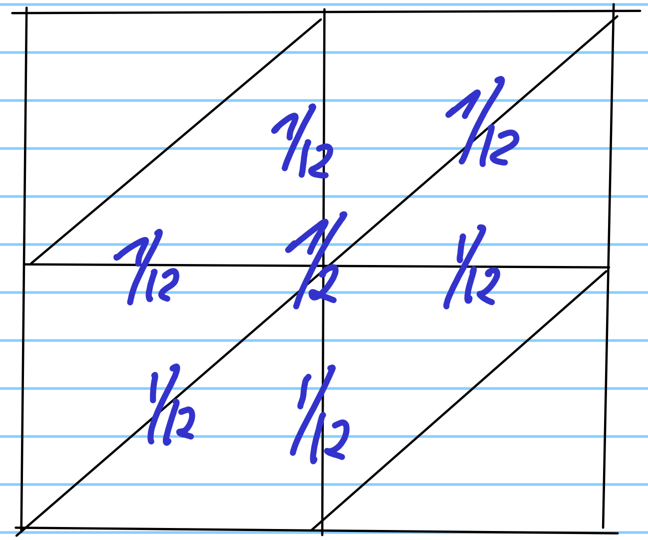
Stencil notation

$$\mathbf{A}_K = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \tag{3.7.5}$$

Contribution of zero-order term

$$\mathbf{M}_K = \frac{1}{12}|K|\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \qquad |K| = \tfrac{1}{2}h^2 \tag{3.7.6}$$

$h^2 \cdot$



Code!

(4)

g)

Implement a C++ function

```
template< typename FUNCTOR >
Eigen::VectorXd computeLoadVector(unsigned int M, FUNCTOR &&f);
```

that computes the right-hand-side vector $\vec{\varphi} \in \mathbb{R}^N$ for the finite element Galerkin discretization of (3.7.1), using the mesh from Fig. 30 and the finite element space $\mathcal{S}_{1,0}^0(\mathcal{M})$, and assuming lexikographic numbering of global shape functions. The argument `f` passes a functor of type `std::function<double(double,double)>` for the source function f.
Evaluate the local integrals approximately with the 2D local trapezoidal rule [Lecture → Eq. (2.4.6.10)].

↳ node-base

$$(\vec{\varphi})_i = h^2 f(x^i)$$

i)

Show by means of a counterexample that for certain values of the constant $c \in]0,1[$ the finite element discretization of (3.7.1) using the mesh from Fig. 30 with $M = 4$ and the finite element space $\mathcal{S}_{1,0}^0(\mathcal{M})$ violates the **discrete maximum principle** established in Sub-problem (3-7.c) in the sense that the finite element Galerkin solution $u_h \in \mathcal{S}_{1,0}^0(\mathcal{M})$ can attain values > 0 though $f \leq 0$.

Seek f : $u_h(x^*) > 0$ for some $x^* \in \Omega$

$\Leftrightarrow$ $u_h(x^i) > 0$ for some $x^i \in \mathcal{V}(\mathcal{M})$

If $(A^{-1})_{ij} < 0$ $\;_{i \neq j}$: choose f such that

[A ≙ Galerkin matrix] $\vec{\varphi} = (-\delta_{j\ell})_{\ell=1}^N$

$\triangleright$ $u_h(x^i) = (A^{-1}\vec{\varphi})_i > 0$

j)

We consider the same finite element Galerkin discretization of (3.7.1) on $\mathcal{M}$ as above. However, now we evaluate all occurring local integrals by means of the local 2D trapezoidal rule [Lecture → Eq. (2.4.6.10)]

$$\int_K \phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{1}{3}|K| \sum_{i=1}^{3} \phi(\boldsymbol{a}^i)\,, \quad K \in \mathcal{M}\,.$$

Write a C++ function

```
Eigen::SparseMatrix<double> computeGalerkinMatrixTR(
    unsigned int M, double c);
```

with the same purpose and arguments as `computeGalerkinMatrix()` from Sub-problem (3-7.g), but now employing the local 2D trapezoidal rule.

$\triangleright$ Element mass matrix $M_K^{TR} = 1/6\, h^2 \cdot I$

L^2-inner product part of $a(\cdot,\cdot)$ will only contribute to the diagonal of the Galerkin matrix

k)

Show that for the Galerkin finite element discretization of (3.7.1) on $\Omega =]0,1[^2$ introduced in Sub-problem (3-7.i) the property $u_h(\boldsymbol{x}) \leq 0$ is satisfied for the finite element solution, if $f(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \Omega$.

Indirect : Assume $\max_{x \in \Omega} u_h(x) > 0$

Notation : $\mu_{ij} = u_h(ih, jh)$, $1 \leq i,j \leq M$

↑ maximal value

[LN § 3.7.2.2]

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$$(1-c)(4\mu_{i,j}-\mu_{i-1,j}-\mu_{i+1,j}-\mu_{i,j-1}-\mu_{i,j+1})+ch^2\mu_{i,j}=\phi_{i,j}\le 0\,.$$

by $f \le 0$

$$4\mu_{i,j}=\underbrace{\phi_{i,j}-\frac{ch^2}{1-c}\mu_{i,j}}_{<0}+(\mu_{i-1,j}+\mu_{i+1,j}+\mu_{i,j-1}+\mu_{i,j+1})\,.$$

since $\mu_{i,j} > 0$

$$\Rightarrow \quad \mu_{i,j}<\underbrace{\frac{1}{4}(\mu_{i-1,j}+\mu_{i+1,j}+\mu_{i,j-1}+\mu_{i,j+1})}_{\text{Average}}\,.$$

$\lightning$ contradicts assumption that $\mu_{i,j}$ is the maximal value.