ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial
Problem 1-1: Quadratic Functionals

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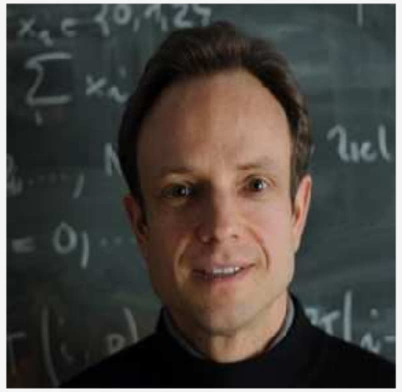

Quadratic functional $\quad J : \begin{cases} V_0 \to \mathbb{R} \\ v \to \tfrac{1}{2}a(v,v) - \ell(v) + c \end{cases}$

symmetric b.l.f. $\qquad$ linear form

a) $$J(v) = \int_{-\pi}^{\pi} x^2 |v(x) + 1|^2 \,\mathrm{d}x\,, \quad v \in C^0_{\mathrm{pw}}([-\pi,\pi])\,. \tag{1.1.2}$$

Trick: Binomial formula

$$J(v) = \underbrace{\int_{-\pi}^{\pi} x^2 |v(x)|^2 dx}_{\frac{1}{2}a(v,v)} + \underbrace{\int_{-\pi}^{\pi} 2x^2 v(x) dx}_{-\ell(v)} + \underbrace{\int_{-\pi}^{\pi} x^2 dx}_{c}$$

$$a(u,v) = 2\int_{-\pi}^{\pi} x^2 u(x)\, v(x) \,\mathrm{d}x\,, \quad u,v \in C^0_{\mathrm{pw}}([-\pi,\pi])\,,$$

$$\ell(v) = -2\int_{-\pi}^{\pi} x^2 v(x) \,\mathrm{d}x\,,$$

- $a(v,v) \geq 0$ clear (quadratic integrand)
- $a(v,v) = 0 \iff \int_{-\pi}^{\pi} x^2 |v(x)|^2 dx = 0 \overset{v \in C^0_{pw}}{\Longrightarrow} v \equiv 0$

s.p.d. $\qquad$ > 0 except in $x = 0$

b) $$J(v) = \int_{-1}^{1} \frac{d^2v}{dx^2}(x) \frac{dv}{dx}(x) \,\mathrm{d}x\,, \quad v \in C^2([-1,1])\,. \tag{1.1.3}$$

$$a(u,v) = \int_{-1}^{1} \frac{d^2u}{dx^2}(x) \frac{dv}{dx}(x) + \frac{d^2v}{dx^2}(x) \frac{du}{dx}(x) \,\mathrm{d}x\,,$$

$$\ell(v) = 0\,.$$

2 Let's find v : $a(v,v) < 0$

Let's try a quadratic polynomial

$$v(x) = \tfrac{1}{2}(x+1)^2 \Rightarrow v'(x) = x-1 \,, \quad v''(x) = 1$$

$$\triangleright \quad a(v,v) = +2$$

c)
$$J(v) = \sum_{j=1}^{n}(v_j - v_{j-1})^2 \,, \quad v = [v_1, \dots, v_n]^\top \in \mathbb{R}^n \,, \quad n \in \mathbb{N} \,, \qquad (1.1.4)$$

with the convention $v_0 := 0$.

$$J(\underline{v}) = \sum_{j=2}^{n} (v_j - v_{j-1})^2 + v_1^2 = \tfrac{1}{2} a(v,v) \geq 0$$

$$\mathsf{a}(\boldsymbol{u}, \boldsymbol{v}) = 2 \sum_{j=1}^{n} (u_j - u_{j-1})(v_j - v_{j-1}) \,,$$

$$\ell(\boldsymbol{v}) = 0 \,.$$

$$a(v,v) = 0 \Rightarrow v_1 = 0 \Rightarrow v_2 = 0 \Rightarrow \dots \Rightarrow v_n = 0$$

$\hookrightarrow$ s.p.d.

Canonical form: $J(\underline{v}) = \frac{1}{2}\underline{v}^T A \underline{v} + \underline{b} \cdot \underline{v} + c$

here $\underline{b} = 0$

Idea: Binomial formula

$$J(\boldsymbol{v}) = v_n^2 + 2\sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n-1} x_{j+1}x_j - \sum_{j=2}^{n} x_j x_{j-1} \,.$$

$$= \tfrac{1}{2} \underline{v}^T A \underline{v}$$

$$= \underline{v}^T \underbrace{\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}}_{= \frac{1}{2} A \in \mathbb{R}^{n,n}} \underline{v}$$

d)
$$J(v) = \int_\Omega |\mathbf{f}(\boldsymbol{x}) \cdot \mathbf{grad}\, u - 1|^2 \, \mathrm{d}\boldsymbol{x} \,, \quad v \in C^1_{\mathrm{pw}}(\Omega) \,, \qquad (1.1.6)$$

Trick: Binomial formula

$$\mathsf{a}(u,v) = 2\int_\Omega (\mathbf{f}(\boldsymbol{x}) \cdot \mathbf{grad}\, u)(\mathbf{f}(\boldsymbol{x}) \cdot \mathbf{grad}\, v) \, \mathrm{d}\boldsymbol{x} \,,$$

$$\ell(v) = -2 \int_\Omega \mathbf{f}(\boldsymbol{x}) \cdot \mathbf{grad}\, v \, \mathrm{d}\boldsymbol{x} \,,$$

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e) $$J(v) = \left(\int_\Omega v(\mathbf{x}) + 1\,\mathrm{d}\mathbf{x}\right)^2 , \quad v \in C^0_{\mathrm{pw}}(\Omega) . \tag{1.1.8}$$

Trick: Binomial formula

$$J(v) = \left(\int_\Omega v(x)\,dx\right)^2 + \int_\Omega v(x)\,dx \cdot \int_\Omega 1\,dx + \left(\int_\Omega 1\,dx\right)^2$$

$$\downarrow \qquad\qquad \downarrow$$

$$a(u,v) = \int_\Omega u(x)\,dx \cdot \int_\Omega v(x)\,dx \qquad \ell(v) = -|\Omega| \int_\Omega v(x)\,dx$$

$a(v,v) \geq 0$

but not s.p.d., because $a(v,v) = 0$, if v has vanishing mean!

g)

Lemma 1.1.9. Extracting the bilinear form from a quadratic functional

*Let J be a quadratic functional on a real vector space V_0 according to [Lecture → **??**],*

$$J(v) = \tfrac{1}{2}\mathsf{a}(v,v) - \ell(v) + c , \quad v \in V_0 , \tag{1.1.10}$$

with a symmetric bilinear form a, a linear form ℓ, and $c \in \mathbb{R}$. Then

$$\mathsf{a}(u,v) = J(u+v) - J(u) - J(v) + J(0) . \tag{1.1.11}$$

▷ Proof by straightforward computation

$$J(u+v) - J(u) - J(v) + J(0)$$
$$= \tfrac{1}{2}\mathsf{a}(u+v,u+v) - \tfrac{1}{2}\mathsf{a}(u,u) - \tfrac{1}{2}\mathsf{a}(v,v) + \underbrace{\ell(u+v) - \ell(u) - \ell(v)}_{=0} + \underbrace{c - c - c + c}_{=0}$$
$$= \cancel{\tfrac{1}{2}\mathsf{a}(u,u)} + \mathsf{a}(u,v) + \cancel{\tfrac{1}{2}\mathsf{a}(v,v)} - \cancel{\tfrac{1}{2}\mathsf{a}(u,u)} - \cancel{\tfrac{1}{2}\mathsf{a}(v,v)} .$$