ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial
Problem 3-5: Error Estimate for Traces

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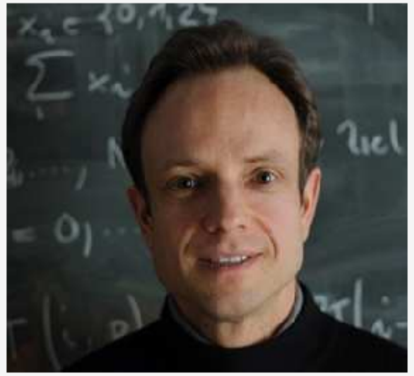

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(C) Seminar für Angewandte Mathematik, ETH Zürich

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \boldsymbol{n}\cdot\mathbf{grad}\, u + u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.5.1}$$

↳ impedance / Robin b.c.

(a) LVP

i.b.p.

$$-\int_\Omega v\Delta u \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{grad}\, u\cdot\mathbf{grad}\, v\,\mathrm{d}\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{n}\cdot\mathbf{grad}\, u) v\,\mathrm{d}S(\boldsymbol{x})$$
$$= \int_\Omega \mathbf{grad}\, u\cdot\mathbf{grad}\, v\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} uv\,\mathrm{d}S(\boldsymbol{x}).$$

$u\in H^1(\Omega)$:

$$\underbrace{\int_\Omega \mathbf{grad}\, u\cdot\mathbf{grad}\, v\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} uv\,\mathrm{d}S(\boldsymbol{x})}_{a(u,v)} = \underbrace{\int_\Omega fv\,\mathrm{d}\boldsymbol{x}}_{\ell(v)} \quad \forall v\in H^1(\Omega) \tag{3.5.2}$$

b) Show uniqueness

- $a(u,u) = \|\mathbf{grad}\, u\|_{L^2}^2 + \|u\|_{L^2(\partial\Omega)}^2 \geq 0$
- $a(u,u) = 0 \Rightarrow \mathbf{grad}\, u = 0 \Rightarrow u \equiv \text{const}$

$\Downarrow$

$\Rightarrow u_{|\partial\Omega} = 0 \Rightarrow u \equiv 0$

$\triangleright$ $a(\cdot,\cdot)$ s.p.d. $\Rightarrow$ uniqueness of u.

2 c) Continuity of $\ell(v)$ w.r.t. $\|\cdot\|_a$

$$\|v\|_a^2 := \|\mathbf{grad}\, v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\partial\Omega)}^2 . \qquad (3.5.9)$$

$|\ell(v)| = \left|\int_\Omega f v \, dx\right| \leq \|f\|_0 \underbrace{\|v\|_0}_{\leq C \|v\|_a}$ by C.S.

To show: $|\ell(v)| \leq \underbrace{C}_{\text{indep. of } v} \|v\|_a \quad \forall v \in H^1(\Omega)$

Trick: $v = \bar{v} + v_*$, $\quad \bar{v} := \frac{1}{|\Omega|}\int_\Omega v \, dx$

[L^2-orthogonal] $\qquad v_* \in H^1_*(\Omega)$

Theorem 1.8.0.20. Second Poincaré-Friedrichs inequality

If $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is bounded, then

$$\exists C = C(\Omega) > 0: \quad \|u\|_0 \leq C \operatorname{diam}(\Omega) \|\mathbf{grad}\, u\|_0 \quad \forall u \in H^1_*(\Omega) .$$

$$\|v\|_0^2 = \|\bar{v}\|_0^2 + \|v_*\|_0^2$$
$$\leq C\left(\|\bar{v}\|_{L^2(\partial\Omega)}^2 + |v_*|_1^2\right)$$
$$\leq C\left(\|v - v_*\|_a^2 + \|v_*\|_a^2\right)$$
Δ-Inequ. $\leq C\left(\|v_*\|_a + \|v\|_a\right)^2$
$$\leq C\|v\|_a^2$$

Theorem 1.9.0.10. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \|u\|_{L^2(\partial\Omega)}^2 \leq C \|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) .$$

$\|v_*\|_a^2 = |v_*|_{H^1(\Omega)}^2 + \|v_*\|_{L^2(\partial\Omega)}^2$ by def. $\|\cdot\|_a$

$$\overset{\text{[Lecture} \to \text{Thm. 1.9.0.10]}}{\leq} |v_*|_{H^1(\Omega)}^2 + C\|v_*\|_{L^2(\Omega)} \|v_*\|_{H^1(\Omega)} \qquad (3.5.7)$$

$$\overset{\text{[Lecture} \to \text{Thm. 1.8.0.20]}}{\leq} C' |v_*|_{H^1(\Omega)}^2 \leq C' \|v\|_a^2 .$$

$|\ell(v)| \leq \underbrace{\|f\|_0 \, C}_{\text{indep. of } v} \|v\|_a \quad \forall v \in H^1(\Omega)$

$-\Delta u = f$

Theorem 3.5.8. Elliptic lifting theorem for convex domains

If $\Omega \subset \mathbb{R}^d$ is convex, $u \in H^1(\Omega)$, Δu f $\in L^2(\Omega)$, $\mathbf{grad}\, u \cdot \boldsymbol{n} + u = 0$ on $\partial\Omega$, then

$$u \in H^2(\Omega) \quad \text{and} \quad \exists C = C(\Omega) > 0: \quad \|u\|_{H^2(\Omega)} \leq C\|\Delta u\ f\|_{L^2(\Omega)} .$$

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d, u solves BVP & Ω convex

$$-\Delta u = f$$

Thm. $\Longrightarrow$ $u \in H^2(\Omega)$, $\|u\|_{H^2(\Omega)} \leq C \|f\|_0$

e, [h-refinement setting]

Based on [Lecture → Thm. 3.3.2.21], qualitatively and quantitatively predict the asymptotic behavior of the error norm $|u - u_i|_{H^1(\Omega)}$ in terms of $N_i := \dim \mathcal{S}_1^0(\mathcal{M}_i)$ for $N_i \to \infty$.

$$\|v\|_a^2 := \|\mathbf{grad}\, v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\partial\Omega)} \,. \qquad (3.5.9)$$

By optimality of Galerkin solutions:

$$\underbrace{\|\mathbf{grad}(u - u_i)\|^2_{L^2(\Omega)} + \|u - u_i\|^2_{L^2(\partial\Omega)}}_{= \|u - u_i\|_a} \leq \underbrace{\|\mathbf{grad}(u - \mathrm{I}_1 u)\|^2_{L^2(\Omega)} + \|u - \mathrm{I}_1 u\|^2_{L^2(\partial\Omega)}}_{\|u - \mathrm{I}_1 u\|_a},$$

$\mathrm{I}_1 u$: nodal interpolant $\in \mathcal{S}_1^0(\mathcal{M})$

Theorem 3.3.2.21. Error estimate for piecewise linear interpolation

For any $u \in C^2(\bar{\Omega})$ and 2D piecewise linear interpolation $\mathrm{I}_1 : C^0(\overline{\Omega}) \to \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh, holds

$$\|u - \mathrm{I}_1 u\|_{L^2(\Omega)} \leq \sqrt{\tfrac{3}{8}}\, h^2_{\mathcal{M}} \big\| \|D^2 u\|_F \big\|_{L^2(\Omega)} \leq C h_m^2 \|u\|_{H^2}$$

$$\|\mathbf{grad}(u - \mathrm{I}_1 u)\|_{L^2(\Omega)} \leq \sqrt{\tfrac{3}{24}}\, \rho_{\mathcal{M}} h_{\mathcal{M}} \big\| \|D^2 u\|_F \big\|_{L^2(\Omega)} \leq C h_m \|u\|_{H^2}$$

where $h_{\mathcal{M}}$ denotes the mesh width (→ Def. 3.2.1.4) and $\rho_{\mathcal{M}}$ the shape regularity measure (→ Def. 3.3.2.20) of $\mathcal{M}$.

Theorem 1.9.0.10. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \leq C \|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,.$$

$$\|u - \mathrm{I}_1 u\|^2_{L^2(\partial\Omega)} \leq C \underbrace{\|u - \mathrm{I}_1 u\|_{L^2(\Omega)}}_{\leq C h_m^2 \|u\|_{H^2}} \underbrace{\|\mathbf{grad}(u - \mathrm{I}_1 u)\|_{L^2(\Omega)}}_{\leq C h_m \|u\|_{H^2}} \quad \text{with} \quad C = C(\Omega) > 0\,.$$

$$\|\mathbf{grad}(u - u_i)\|^2_{L^2(\Omega)} + \underbrace{\cancel{\|u - u_i\|^2_{L^2(\partial\Omega)}}}_{\geq 0} \leq \underbrace{\underbrace{\|\mathbf{grad}(u - \mathrm{I}_1 u)\|^2_{L^2(\Omega)}}_{O(h_m^2)} + \underbrace{\|u - \mathrm{I}_1 u\|^2_{L^2(\partial\Omega)}}_{O(h_m^3)}}_{O(h_m^2)},$$

Reg. ref: $N_i \approx h_i^{-2}$

$$\|u - u_i\|_{H^1(\Omega)} \leq C_2 N_i^{-1/2} |u|_{H^2(\Omega)} = O(N_i^{-1/2})\,.$$

for $N_i \to \infty$

(4) f)

Employ the duality techniques introduced in [Lecture → § 3.6.3.3] to derive a sharp asymptotic estimate for the error norm $\|u-u_i\|_{L^2(\Omega)}$ as $N_i := \dim \mathcal{S}_1^0(\mathcal{M}_i) \to \infty$.

Theorem 3.6.1.7. Duality estimate for linear functional output

Define the dual solution $g_F \in V_0$ *to* F *as solution of the* dual variational problem

$$g_F \in V_0:\quad a(v, g_F) = F(v) \quad \forall v \in V_0 .$$

Then [F ≙ linear continuous output functional]

$$|F(u) - F(u_h)| \leq \|u-u_h\|_a \inf_{v_h \in V_{0,h}} \|g_F - v_h\|_a . \tag{3.6.1.8}$$

Here : $F(v) = \int_\Omega (u-u_i)\, v \, dx$

$g \in H^1(\Omega)$

$$\int_\Omega \mathbf{grad}\, g \cdot \mathbf{grad}\, v \,dx + \int_{\partial\Omega} g\, v \,dS = \int_\Omega (u-u_i) v \,dx \quad \forall v \in H^1(\Omega) . \tag{3.5.12}$$

⇓ Strong form

$$-\Delta g = u - u_i \ \text{ in } \Omega \ , \quad \boldsymbol{n}\cdot \mathbf{grad}\, g + g = 0 \ \text{ on } \partial\Omega , \tag{3.5.11}$$

Theorem 3.5.8. Elliptic lifting theorem for convex domains

If $\Omega \subset \mathbb{R}^d$ *is convex*, $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$, $\mathbf{grad}\, u \cdot \boldsymbol{n} + u = 0$ *on* $\partial\Omega$, *then*

$$u \in H^2(\Omega) \quad \text{and} \quad \exists C = C(\Omega) > 0:\quad \|u\|_{H^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} .$$

▷ $g \in H^2(\Omega)$, $\|g\|_{H^2} \leq C \|u-u_i\|_0$

$\|u-u_i\|_0^2 = F(u-u_i) = a(u-u_i, g)$

by Galerkin orthogonality $= a(u-u_i, g-v_i) \quad \forall v_i \in S_1^0(M)$

$$\|u-u_i\|^2_{L^2(\Omega)} = \int_\Omega \mathbf{grad}(g-v_i)\cdot \mathbf{grad}(u-u_i)\,dx + \int_{\partial\Omega} (g-v_i)(u-u_i)\,dS$$

C.S. for $a(\cdot,\cdot)$

$$\leq \left(\|\mathbf{grad}(g-v_i)\|^2_{L^2(\Omega)} + \|g-v_i\|^2_{L^2(\partial\Omega)}\right)^{\frac12} \cdot \left(\|\mathbf{grad}(u-u_i)\|^2_{L^2(\Omega)} + \|u-u_i\|^2_{L^2(\partial\Omega)}\right)^{\frac12} = O(h_M) \text{ from g)}$$

replace v_i with $I_i g$ & use arguments from g),

$\leq C h_M \|g\|_{H^2} \leq C h_M \|u-u_i\|_0$

$$\|u-u_i\|_{L^2(\Omega)} \leq C_1 N_i^{-1} |u|_{H^2(\Omega)} = O(N_i^{-1}) = O(h_M^2)$$

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g)

In the setting of the previous sub-problem, what asymptotic behavior can be deduced from the multiplicative trace inequality [Lecture → Thm. 1.9.0.10] for the error norm $\|u-u_i\|_{L^2(\partial\Omega)}$ on the boundary in terms of $N_i := \dim \mathcal{S}_1^0(\mathcal{M}_i) \to \infty$?

Theorem 1.9.0.10. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,.$$

$$\|u-u_i\|^2_{L^2(\partial\Omega)} \leq C_3 \underbrace{\|u-u_i\|_{L^2(\Omega)}}_{O(h_m^2)} \underbrace{\|u-u_i\|_{H^1(\Omega)}}_{O(h_m)} \leq C_1C_2C_3N_i^{-3/2}|u|^2_{H^2(\Omega)}.$$

$$\|u-u_i\|_{L^2(\partial\Omega)} \leq C_4 h_i^{3/2}|u|_{H^2(\Omega)} = O(h_i^{3/2}) = O(N_i^{-3/4})$$

(i) $B(v) = \int_{\partial\Omega} v\, dS$

In the file `tee_lapl_robin_assembly.cc` (name due to historical reasons) implement an LEHRFEM++-based function

```
double bdFunctionalEval(
    std::shared_ptr<lf::uscalfe::FeSpaceLagrangeO1<double>>& fe_space,
    Eigen::VectorXd& coeff_vec);
```

that computes $B(v_h) := \int_{\partial\Omega} v_h\, dS(x)$, where $v_h \in \mathcal{S}_1^0(\mathcal{M})$ and its basis expansion coefficients are passed through `coeff_vec`. The argument `fe_space` also passes information about the (planar triangular) mesh $\mathcal{M}$ and the "d.o.f. handler" for the finite element space.

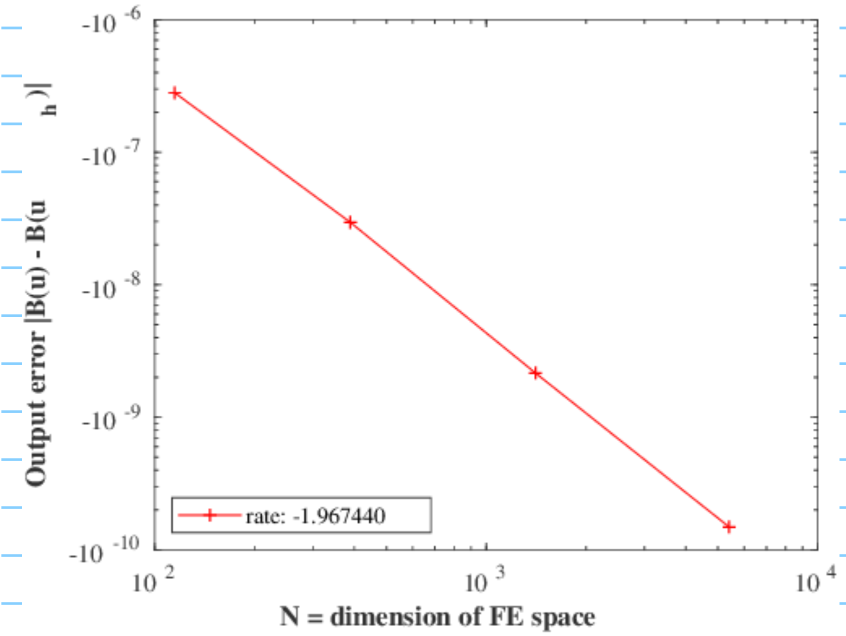
$$\int_{\partial\Omega} v_h\, dS = \sum_{e\subset\partial\Omega} \tfrac{1}{2}|e|\,(v_h(a_e^1)+v(a_e^2))$$

[e ≙ edge]

C++11 code 3.5.15: Sub-problem (3-5.i): function bdFunctionalEval() ➙ GITHUB

```
double bdFunctionalEval(
    std::shared_ptr<lf::uscalfe::FeSpaceLagrangeO1<double>>& fe_space,
    Eigen::VectorXd& coeff_vec) {
  double bd_functional_val = 0;

  /* SOLUTION_BEGIN */
  // Reference to mesh
  std::shared_ptr<const lf::mesh::Mesh> mesh_p{fe_space->Mesh()};
  // Obtain local->global index mapping for current finite element space
  const lf::assemble::DofHandler& dofh{fe_space->LocGlobMap()};

  // Obtain an array of boolean flags for the edges of the mesh: 'true'
  // indicates that the edge lies on the boundary. This predicate will
  // guarantee
  // that the computations are carried only on the edges of the mesh.
  auto bd_flags{lf::mesh::utils::flagEntitiesOnBoundary(mesh_p, 1)};

  // Creating predicate that will guarantee that the computations are carried
  // only on the edges of the mesh using the boundary flags
  auto edges_predicate = [&bd_flags](const lf::mesh::Entity& edge) -> bool {
    return bd_flags(edge);
  };

  // Computing the integral of function_vec on the flagged edges
  double edge_length;
  for (const lf::mesh::Entity& edge : mesh_p->Entities(1)) {
    if (bd_flags(edge)) {
      // Obtain endpoints of the edge
      auto endpoints = lf::geometry::Corners(*(edge.Geometry()));
      LF_ASSERT_MSG(endpoints.cols() == 2, "Wrong no endpoints in " << edge);
      // Compute length of the edge
      edge_length = (endpoints.col(1) - endpoints.col(0)).norm();
      // Find the endpoints global indices
      auto dof_idx = dofh.GlobalDofIndices(edge);
      BOOST_ASSERT(dofh.NoLocalDofs(edge) == 2);
      bd_functional_val +=
          (coeff_vec.coeff(dof_idx[0]) + coeff_vec.coeff(dof_idx[1])) *
          edge_length / 2.0;
    }
  }
  /* SOLUTION_END */
  return bd_functional_val;
}
```

j) $B(u)$ known, track $|B(u)-B(u_i)|$



◁ Alg. cvg. rate 2

k)

Prove that for the linear functional $v \mapsto B(v)$ as defined in (3.5.14) holds

$$B(u) = \int_\Omega f(x)\,\mathrm{d}x \quad \text{and} \quad B(u_h) = \mathbf{1}^\top \vec{\varphi}\,, \qquad (3.5.17)$$

↳ $\hat{=}$ FE solution

where $u \in H^1(\Omega)$ is the exact solution of (3.5.1), $u_h \in \mathcal{S}_1^0(\mathcal{M})$ the finite element solution, and $\vec{\varphi} \in \mathbb{R}^N$, $N := \dim \mathcal{S}_1^0(\mathcal{M})$, is the right-hand-side vector of the Galerkin linear system. The symbol $\mathbf{1}$ stands for the vector $\in \mathbb{R}^N$ with entries all $= 1$.

$$-\int_\Omega v\Delta u\,\mathrm{d}x = \int_\Omega \mathbf{grad}\,u \cdot \mathbf{grad}\,v\,\mathrm{d}x - \int_{\partial\Omega} (\boldsymbol{n} \cdot \mathbf{grad}\,u) v\,\mathrm{dS}(\boldsymbol{x})$$
$$= \int_\Omega \mathbf{grad}\,u \cdot \mathbf{grad}\,v\,\mathrm{d}x + \int_{\partial\Omega} uv\,\mathrm{dS}(\boldsymbol{x})\,.$$

$$\int_\Omega \mathbf{grad}\,u_h \cdot \mathbf{grad}\,v_h\,\mathrm{d}x + \int_{\partial\Omega} u_h v_h\,\mathrm{dS}(\boldsymbol{x}) = \overbrace{\int_\Omega f v_h\,\mathrm{d}x}^{\ell(v_h)} \quad \forall v_h \in \mathcal{S}_1^0(\mathcal{M}) \qquad (3.5.2)$$

• $v \equiv 1$: $\mathbf{grad}\,v \equiv 0 \Rightarrow B(u) = \int_{\partial\Omega} u \cdot 1\,dS = \int_\Omega f\,dx$

• Note $\{x \to 1\} \in \mathcal{S}_1^0(\mathcal{M})$

$1 = \sum_{\ell=1}^N b_h^\ell$

↳ global shape functions

$v_h \equiv 1 \Rightarrow B(u_h) = \int_\Omega f\,dx$ *

$B(u_h) = \ell(1) = \ell\left(\sum_{\ell=1}^N b_h^\ell\right)$

$= \sum_\ell \underbrace{\ell(b_h^\ell)}_{(\vec{\varphi})_\ell} = 1^\top \cdot \vec{\varphi}$

* in the case of exact computation of $\vec{\varphi}$

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e) Error $|B(u_h) - B(u)|$ due to the use of numerical quadrature!