ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 3-5: Error Estimate for Traces

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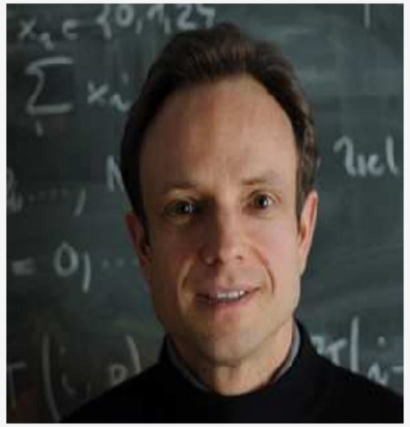

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(C) Seminar für Angewandte Mathematik, ETH Zürich

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \qquad , \ f \in L^2(\Omega) \\
\boldsymbol{n} \cdot \mathbf{grad}\, u + u &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{3.5.1}
$$

_01

↳ impedance / Robin b.c.

a, LVP :

$$
-\int_\Omega v \Delta u \,\mathrm{d}x \overset{\text{i.b.p.}}{=} \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x - \int_{\partial\Omega} (\boldsymbol{n} \cdot \mathbf{grad}\, u) v \,\mathrm{d}S(\boldsymbol{x})
$$

$$
= \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x + \int_{\partial\Omega} uv \,\mathrm{d}S(\boldsymbol{x}) .
$$

_02

$u \in H^1(\Omega)$:

$$
\underbrace{\int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x + \int_{\partial\Omega} uv \,\mathrm{d}S(\boldsymbol{x})}_{= a(u,v)} = \underbrace{\int_\Omega fv \,\mathrm{d}x}_{= \ell(v)} \quad \forall v \in H^1(\Omega) \tag{3.5.2}
$$

b, Show uniqueness :

- $a(u,u) = \|\mathrm{grad}\, u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} \geq 0$
- $a(u,u) = 0 \;\Rightarrow\; \mathrm{grad}\, u = 0 \;\Rightarrow\; u \equiv \text{const}$

$\Rightarrow \; u_{|\partial\Omega} \;\Longrightarrow\; u \equiv 0$

▷ $a(\cdot,\cdot)$ s.p.d. $\Rightarrow$ uniqueness of u.

② c) Continuity of $\ell(v)$ w.r.t. $\|\cdot\|_a$

$$\|v\|_a^2 := \|\mathbf{grad}\, v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\partial\Omega)} . \tag{3.5.9}$$

– 10

C.S. $|\ell(v)| = \int_\Omega f v \, dx \leq \|f\|_0 \|v\|_0$

To show: $\|v\|_0 \leq C \|v\|_a \quad \forall v \in H^1(\Omega)$

Trick: $v = \overline{v} + v_*$, $\quad \overline{v} = \frac{1}{|\Omega|}\int_\Omega v \, dx$

[$L^2(\Omega)$-orthogonal] $\quad v_* \in H^1_*(\Omega)$

Theorem 1.8.0.20. Second Poincaré-Friedrichs inequality PF2

If $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is bounded, then

$$\exists C = C(\Omega) > 0: \quad \|u\|_0 \leq C \operatorname{diam}(\Omega) \|\mathbf{grad}\, u\|_0 \quad \forall u \in H^1_*(\Omega) .$$

$$\begin{aligned}
\|v\|_0^2 &= \|\overline{v}\|_0^2 + \|v_*\|_0^2 \\
&\leq C\left(\|\overline{v}\|^2_{L^2(\Omega)} + |v_*|_{H^1(\Omega)} \right) \\
&\leq C'\left(\|v - v_*\|_a^2 + \|v\|_a^2 \right) \\
&\overset{\Delta\text{-inequ.}}{\leq} C''\left(\|v_*\|_a + \|v\|_a \right)^2 \\
&\leq C^{(4)} \|v\|_a^2
\end{aligned}$$

Theorem 1.9.0.10. Multiplicative trace inequality mti

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \leq C \|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) .$$

$$\begin{aligned}
\|v_*\|_a^2 &= |v_*|^2_{H^1(\Omega)} + \|v_*\|^2_{L^2(\partial\Omega)} \quad \text{by def. of } \|\cdot\|_a \\
&\overset{[\text{Lecture} \to \text{Thm. 1.9.0.10}]}{\leq} |v_*|^2_{H^1(\Omega)} + C\|v_*\|_{L^2(\Omega)} \|v_*\|_{H^1(\Omega)} \\
&\overset{[\text{Lecture} \to \text{Thm. 1.8.0.20}]}{\leq} C'|v_*|^2_{H^1(\Omega)} \leq C'\|v\|_a^2 .
\end{aligned} \tag{3.5.7}$$

– 05

$$\underbrace{\int_\Omega f(\overline{v} + v_*) \, \mathbf{dx}}_{\ell(v)} \leq \|f\|_{L^2(\Omega)} \left(\|\overline{v}\|_{L^2(\Omega)} + \|v_*\|_{L^2(\Omega)} \right)$$

– 06

$$\begin{aligned}
&\overset{(3.5.6)}{\leq} C\|f\|_{L^2(\Omega)} (\|\overline{v}\|_a + \|v_*\|_a) \\
&\overset{\Delta\text{-inequ.}}{\leq} C'C''\|f\|_{L^2(\Omega)} (\|v\|_a + \|v_*\|_a) \overset{(3.5.7)}{\leq} \overbrace{C''\|f\|_{L^2(\Omega)}}^{\text{a constant}} \|v\|_a .
\end{aligned}$$

Theorem 3.5.8. Elliptic lifting theorem for convex domains – 08

If $\Omega \subset \mathbb{R}^d$ is convex, $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$, $\mathbf{grad}\, u \cdot \boldsymbol{n} + u = 0$ on $\partial\Omega$, then

$$u \in H^2(\Omega) \quad \text{and} \quad \exists C = C(\Omega) > 0: \quad \|u\|_{H^2(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)} .$$

d, u solves BVP & Ω convex
↳ $-\Delta u = f$

Thm 3.5.8: $u \in H^2(\Omega)$, $\|u\|_{H^2(\Omega)} \leq C\|f\|_0$

e, [h-refinement setting]

Based on [Lecture → Thm. 3.3.2.21], qualitatively and quantitatively predict the asymptotic behavior of the error norm $|u - u_i|_{H^1(\Omega)}$ in terms of $N_i := \dim \mathcal{S}_1^0(\mathcal{M}_i)$ for $N_i \to \infty$. — 09

$$\|v\|_a^2 := \|\mathbf{grad}\, v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\partial\Omega)}^2 . \qquad (3.5.9)$$ — 10

By optimality of Galerkin solution w.r.t. $\|\cdot\|_a$

$$\|\mathbf{grad}(u-u_i)\|_{L^2(\Omega)}^2 + \|u-u_i\|_{L^2(\partial\Omega)}^2 \leq \|\mathbf{grad}(u-\mathrm{I}_1 u)\|_{L^2(\Omega)}^2 + \|u-\mathrm{I}_1 u\|_{L^2(\partial\Omega)}^2 ,$$ — 11

$\mathrm{I}_1 u \hat{=}$ nodal interpolant

Theorem 3.3.2.21. Error estimate for piecewise linear interpolation [3.3.29]

For any $u \in C^2(\bar{\Omega})$ and 2D piecewise linear interpolation $\mathrm{I}_1 : C^0(\overline{\Omega}) \to \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh, holds

$$\|u - \mathrm{I}_1 u\|_{L^2(\Omega)} \leq \sqrt{\tfrac{3}{8}}\, h_{\mathcal{M}}^2 \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)} ,$$

$$\|\mathbf{grad}(u - \mathrm{I}_1 u)\|_{L^2(\Omega)} \leq \sqrt{\tfrac{3}{24}}\, \rho_{\mathcal{M}} h_{\mathcal{M}} \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)} .$$

where $h_{\mathcal{M}}$ denotes the mesh width (→ Def. 3.2.1.4) and $\rho_{\mathcal{M}}$ the shape regularity measure (→ Def. 3.3.2.20) of $\mathcal{M}$.

by

Theorem 1.9.0.10. Multiplicative trace inequality [1.9.07]

$$\exists C = C(\Omega) > 0: \quad \|u\|_{L^2(\partial\Omega)}^2 \leq C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) .$$

$$\|u - \mathrm{I}_1 u\|_{L^2(\partial\Omega)}^2 \leq C \underbrace{\|u - \mathrm{I}_1 u\|_{L^2(\Omega)}}_{\leq C h_m^2 |u|_{H^2(\Omega)}} \underbrace{\|\mathbf{grad}(u - \mathrm{I}_1 u)\|_{L^2(\Omega)}}_{\leq C h_m |u|_{H^2(\Omega)}} \quad \text{with} \quad C = C(\Omega) > 0 .$$ — 12

$$\underbrace{\|\mathbf{grad}(u-u_i)\|_{L^2(\Omega)}^2 + \|u-u_i\|_{L^2(\partial\Omega)}^2}_{\geq 0} \leq \underbrace{\underbrace{\|\mathbf{grad}(u-\mathrm{I}_1 u)\|_{L^2(\Omega)}^2}_{O(h_m^2)} + \underbrace{\|u-\mathrm{I}_1 u\|_{L^2(\partial\Omega)}^2}_{O(h_m^3)}}_{O(h_m^2)} ,$$ — 11

Reg. ref: $N_i \approx h_m^{-2}$

$$\|u - u_i\|_{H^1(\Omega)} \leq C_2 N_i^{-1/2} |u|_{H^2(\Omega)} = O(N_i^{-1/2}) .$$ — 14

(4) F_i

Employ the duality techniques introduced in [Lecture → § 3.6.3.3] to derive a sharp asymptotic estimate for the error norm $\|u-u_i\|_{L^2(\Omega)}$ as $N_i := \dim \mathcal{S}_1^0(\mathcal{M}_i) \to \infty$. – 15

Theorem 3.6.1.7. Duality estimate for linear functional output [3.6.11]

Define the dual solution $g_F \in V_0$ to F as solution of the dual variational problem

$$g_F \in V_0:\quad \mathsf{a}(v, g_F) = F(v) \quad \forall v \in V_0 .$$

Then

$$|F(u) - F(u_h)| \leq \|u-u_h\|_{\mathsf{a}} \inf_{v_h \in V_{0,h}} \|g_F - v_h\|_{\mathsf{a}} . \tag{3.6.1.8}$$

Here: $F(v) = \int_\Omega (u-u_i)\, v \,dx$

$g \in H^1(\Omega)$

$$\int_\Omega \mathbf{grad}\, g \cdot \mathbf{grad}\, v \,\mathrm{d}x + \int_{\partial\Omega} g\, v \,\mathrm{d}S = \int_\Omega (u-u_i) v \,\mathrm{d}x \quad \forall v \in H^1(\Omega) . \quad \text{– 17} \tag{3.5.12}$$

$$-\Delta g = u - u_i \ \text{ in } \Omega \ , \quad \boldsymbol{n}\cdot \mathbf{grad}\, g + g = 0 \ \text{ on } \partial\Omega , \quad \text{– 16} \tag{3.5.11}$$

Theorem 3.5.8. Elliptic lifting theorem for convex domains – 08

If $\Omega \subset \mathbb{R}^d$ is convex, $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$, $\mathbf{grad}\, u \cdot \boldsymbol{n} + u = 0$ on $\partial\Omega$, then

$$u \in H^2(\Omega) \quad \text{and} \quad \exists C = C(\Omega) > 0:\quad \|u\|_{H^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} .$$

$\triangleright$ $g \in H^2(\Omega)$, $\|g\|_{H^2} \leq C \|u - u_i\|_0$

$$\|u-u_i\|_0^2 = F(u-u_i)$$
$$= a(u-u_i, g)$$
$$\text{G.O.} \quad = a(u-u_i, g - v_i) \quad \forall v_i \in \mathcal{S}_1^0(\mathcal{M}_i)$$

$$\|u-u_i\|^{\cancel{2}}_{L^2(\Omega)} = \int_\Omega \mathbf{grad}(g-v_i)\cdot \mathbf{grad}(u-u_i)\,\mathrm{d}x + \int_{\partial\Omega} (g-v_i)(u-u_i)\,\mathrm{d}S$$

C.S. for $a(\cdot,\cdot)$

$$\leq \left(\|\mathbf{grad}(g-v_i)\|^2_{L^2(\Omega)} + \|g - v_i\|^2_{L^2(\partial\Omega)} \right)^{\frac12} \cdot \quad = O(h_m)$$
$$\left(\|\mathbf{grad}(u-u_i)\|^2_{L^2(\Omega)} + \|u - u_i\|^2_{L^2(\partial\Omega)} \right)^{\frac12} \quad = O(h_m)$$

[from g_i]

replace $v_i \leftarrow I_1 g$ & use + argument from g_i

$$\leq C h_m |g|_{H^2(\Omega)} \leq C h_m \cancel{\|u-u_i\|_0}$$

Theorem 3.3.2.21. Error estimate for piecewise linear interpolation [3.3.29]

For any $u \in C^2(\bar\Omega)$ and 2D piecewise linear interpolation $\mathsf{I}_1 : C^0(\overline{\Omega}) \to \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh, holds

$$\|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \leq \sqrt{\tfrac{3}{8}}\, h_{\mathcal{M}}^2 \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)} ,$$
$$\|\mathbf{grad}(u - \mathsf{I}_1 u)\|_{L^2(\Omega)} \leq \sqrt{\tfrac{3}{24}}\, \rho_{\mathcal{M}} h_{\mathcal{M}} \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)} .$$

where $h_{\mathcal{M}}$ denotes the mesh width (→ Def. 3.2.1.4) and $\rho_{\mathcal{M}}$ the shape regularity measure (→ Def. 3.3.2.20) of $\mathcal{M}$.

$$\|u-u_i\|_{L^2(\Omega)} \leq C_1 N_i^{-1} |u|_{H^2(\Omega)} = O(N_i^{-1}) . \quad \text{– 20}$$

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g)

In the setting of the previous sub-problem, what asymptotic behavior can be deduced from the multiplicative trace inequality [Lecture → Thm. 1.9.0.10] for the error norm $\|u-u_i\|_{L^2(\partial\Omega)}$ on the boundary in terms of $N_i := \dim \mathcal{S}_1^0(\mathcal{M}_i) \to \infty$?

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Theorem 1.9.0.10. Multiplicative trace inequality [1.9.0.7]

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \le C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,.$$

$$\|u-u_i\|^2_{L^2(\partial\Omega)} \le C_3 \underbrace{\|u-u_i\|_{L^2(\Omega)}}_{O(h_{\mathcal{M}}^2)} \underbrace{\|u-u_i\|_{H^1(\Omega)}}_{O(h_{\mathcal{M}})} \le C_1C_2C_3N_i^{-3/2}|u|^2_{H^2(\Omega)}.$$

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$$\|u-u_i\|_{L^2(\partial\Omega)} \le C_4 h_i^{3/2}|u|_{H^2(\Omega)} = O(h_i^{3/2}) = O(N_i^{-3/4})$$

$$N_i \approx h_{\mathcal{M}_i}^{-2}$$

$$B(v) = \int_{\Omega} v \, dS$$

(i)

In the file `tee_lapl_robin_assembly.cc` (name due to historical reasons) implement an LehrFEM++-based function

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```
double bdFunctionalEval(
    std::shared_ptr<lf::uscalfe::FeSpaceLagrangeO1<double>>& fe_space,
    Eigen::VectorXd& coeff_vec);
```

that computes $B(v_h) := \int_{\partial\Omega} v_h \, dS(x)$, where $v_h \in \mathcal{S}_1^0(\mathcal{M})$ and its basis expansion coefficients are passed through `coeff_vec`. The argument `fe_space` also passes information about the (planar triangular) mesh $\mathcal{M}$ and the "d.o.f. handler" for the finite element space.

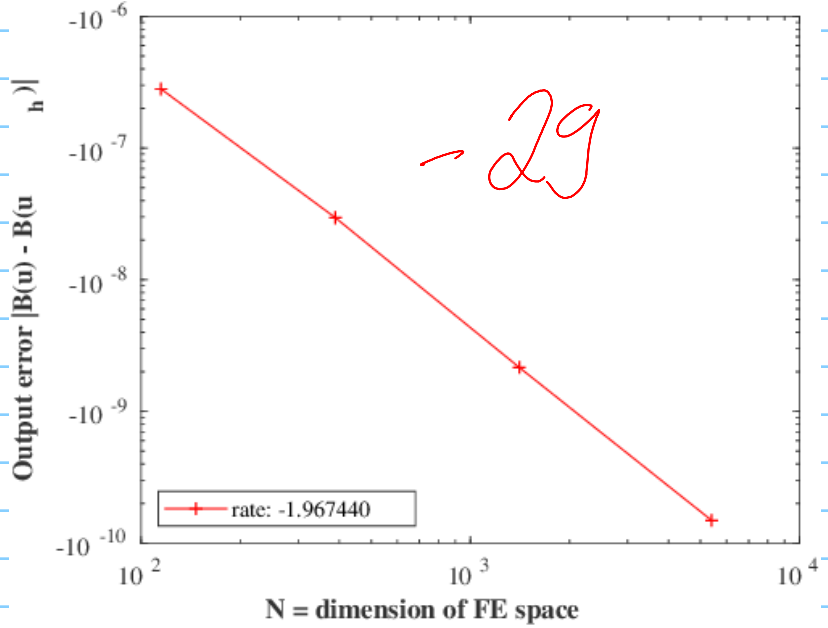
$$\int_{\partial\Omega} v_h \, dS = \sum_{\substack{e\subset\partial\Omega \\ [e \,\hat{=}\, \text{edge}]}} \tfrac{1}{2}|e|\left(v_h(a_e^1) + v_h(a_e^2)\right)$$

C++11 code 3.5.15: Sub-problem (3-5.i): function `bdFunctionalEval()` ⇢ GITHUB

−26

```
double bdFunctionalEval(
    std::shared_ptr<lf::uscalfe::FeSpaceLagrangeO1<double>>& fe_space,
    Eigen::VectorXd& coeff_vec) {
  double bd_functional_val = 0;

  /* SOLUTION_BEGIN */
  // Reference to mesh
  std::shared_ptr<const lf::mesh::Mesh> mesh_p{fe_space->Mesh()};
  // Obtain local->global index mapping for current finite element space
  const lf::assemble::DofHandler& dofh{fe_space->LocGlobMap()};

  // Obtain an array of boolean flags for the edges of the mesh: 'true'
  // indicates that the edge lies on the boundary. This predicate will
  // guarantee
  // that the computations are carried only on the edges of the mesh.
  auto bd_flags{lf::mesh::utils::flagEntitiesOnBoundary(mesh_p, 1)};

  // Creating predicate that will guarantee that the computations are carried
  // only on the edges of the mesh using the boundary flags
  auto edges_predicate = [&bd_flags](const lf::mesh::Entity& edge) -> bool {
    return bd_flags(edge);
  };

  // Computing the integral of function_vec on the flagged edges
  double edge_length;
  for (const lf::mesh::Entity& edge : mesh_p->Entities(1)) {
    if (bd_flags(edge)) {
      // Obtain endpoints of the edge
      auto endpoints = lf::geometry::Corners(*(edge.Geometry()));
      LF_ASSERT_MSG(endpoints.cols() == 2, "Wrong no endpoints in " << edge);
      // Compute length of the edge
      edge_length = (endpoints.col(1) - endpoints.col(0)).norm();
      // Find the endpoints global indices
      auto dof_idx = dofh.GlobalDofIndices(edge);
      BOOST_ASSERT(dofh.NoLocalDofs(edge) == 2);
      bd_functional_val +=
          (coeff_vec.coeff(dof_idx[0]) + coeff_vec.coeff(dof_idx[1])) *
          edge_length / 2.0;
    }
  }
  /* SOLUTION_END */
  return bd_functional_val;
}
```

6 j, $B(u)$ known, track $|B(u)-B(u_i)|$



▷ Alg. cvg. with rate 2

k, Prove that for the linear functional $v \mapsto B(v)$ as defined in (3.5.14) holds

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$$\int_{\partial\Omega} u\,dS = B(u) = \int_\Omega f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{and} \quad B(u_h) = \mathbf{1}^\top \vec{\boldsymbol{\varphi}}\,, \tag{3.5.17}$$

where $u \in H^1(\Omega)$ is the exact solution of (3.5.1), $u_h \in \mathcal{S}_1^0(\mathcal{M})$ the finite element solution, and $\vec{\boldsymbol{\varphi}} \in \mathbb{R}^N$, $N := \dim \mathcal{S}_1^0(\mathcal{M})$, is the right-hand-side vector of the Galerkin linear system. The symbol $\mathbf{1}$ stands for the vector $\in \mathbb{R}^N$ with entries all $= 1$.

$$\int_\Omega \operatorname{grad} u_h \cdot \operatorname{grad} v_h\,dx + \int_{\partial\Omega} u_h v_h\,dS = \overbrace{\int_\Omega f(x)\,v_h(x)\,dx}^{\ell_h(v_h)} \quad \forall v \in H^1(\Omega) \quad \forall v_h \in \mathcal{S}_1^0(\mathcal{M})$$

$v \equiv 1 \in \mathcal{S}_1^0(\mathcal{M})$: $\quad \int_{\partial\Omega} u\,d = \int_\Omega f\,dx$

As $\{x \mapsto 1\} \in \mathcal{S}_1^0(\mathcal{M})$

$$1 = \sum_{\ell=1}^N b_h^\ell$$

$v_h \equiv 1 \Rightarrow$

$$B(u_h) = \int u_h\,dS = \ell_h(1) = \sum_{\ell=1}^N \ell_h(b_h^\ell) = \sum_{\ell=1}^N (\vec{\varphi})_\ell$$

due to numerical quadrature

ℓ, If $\ell = \ell_h \Rightarrow B(u) = B(u_h)$

due to quadrature error in the computation of $\vec{\varphi}$!