1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 2-2: Transformation of Galerkin Matrices

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Date: March 2, 2019

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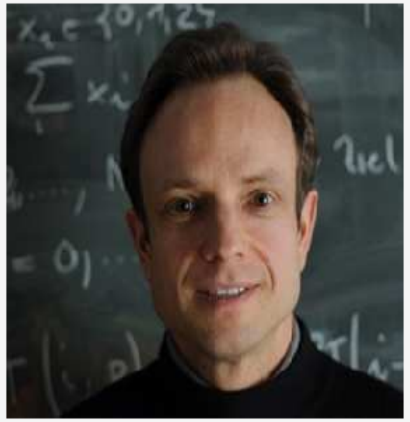

Prerequisites : Section 2.2

Abstract perspective

LVP : $u \in V_0$: $a(u,v) = \ell(v)$ $\forall v \in V_0$

Galerkin trial/test space [Step I] : $V_{0,h} \subset V_0$, $N := \dim V_{0,h} < \infty$

Step II : Basis $\mathfrak{B}_h = \{b_h^1, \dots, b_h^N\}$

$\hookrightarrow$ Galerkin matrix $\mathbf{A} = [a(b_h^j, b_h^i)]_{i,j=1}^N$

Lemma 2.2.3.2. Effect of change of basis on Galerkin matrix

Consider (2.2.1.1) *and two bases of* $V_{0,h}$,

"old" basis $\mathfrak{B}_h := \{b_h^1, \dots, b_h^N\}$, $\widetilde{\mathfrak{B}}_h := \{\widetilde{b}_h^1, \dots, \widetilde{b}_h^N\}$, "new" basis

related by the basis transformation matrix $\mathbf{S}$ *according to*

$$\widetilde{b}_h^j = \sum_{k=1}^N s_{jk} b_h^k \quad \text{with} \quad \mathbf{S} = [s_{jk}]_{j,k=1}^N \in \mathbb{R}^{N,N} \text{ regular.} \tag{2.2.3.3}$$

Then the Galerkin matrices $\mathbf{A}, \widetilde{\mathbf{A}} \in \mathbb{R}^{N,N}$, *the right hand side vectors* $\vec{\boldsymbol{\varphi}}, \widetilde{\vec{\boldsymbol{\varphi}}} \in \mathbb{R}^N$, *and the coefficient vectors* $\vec{\boldsymbol{\mu}}, \widetilde{\vec{\boldsymbol{\mu}}} \in \mathbb{R}^N$, *respectively, satisfy*

$$\widetilde{\mathbf{A}} = \mathbf{S}\mathbf{A}\mathbf{S}^\top \;,\quad \widetilde{\vec{\boldsymbol{\varphi}}} = \mathbf{S}\vec{\boldsymbol{\varphi}} \;,\quad \widetilde{\vec{\boldsymbol{\mu}}} = \mathbf{S}^{-\top}\vec{\boldsymbol{\mu}} \;. \tag{2.2.3.4}$$

New basis : ($N = 2M$, $M \in \mathbb{N}$)

$$\begin{aligned} \widetilde{b}_h^j &:= b_h^{2j-1} + b_h^{2j} && \text{for } j \in \{1, \dots, M\}, \\ \widetilde{b}_h^j &:= b_h^{2(j-M)-1} - b_h^{2(j-M)} && \text{for } j \in \{M+1, \dots, N\}. \end{aligned} \tag{2.2.2}$$

② a) $N=4 \Leftrightarrow M=2$

(2.2.3.3) $\Rightarrow$

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

$S^T S = 2I$

b)

$$\mathbf{S} = \left[\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & \dots & & & & \dots & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & & & & \vdots \\ 0 & 0 & 0 & 0 & 1 & 1 & & & & \\ \vdots & & & & & & \ddots & & & \vdots \\ & & & & & & 1 & 1 & 0 & 0 \\ 0 & \dots & & & & \dots & 0 & 0 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 & \dots & & & & \dots & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & & & & \vdots \\ 0 & 0 & 0 & 0 & 1 & -1 & & & & \\ \vdots & & & & & & \ddots & & & \vdots \\ & & & & & & 1 & -1 & 0 & 0 \\ 0 & \dots & & & & \dots & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} M \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} M \end{array}.$$

$S^T S = 2I \Rightarrow S^{-1} = \frac{1}{2} S^T$

c) Concretization of $\tilde{A} = SAS^T$

Every entry $(\tilde{\mathbf{A}})_{ij}$, $i,j \in \{1,\dots,N\}$, of $\tilde{\mathbf{A}}$ can be obtained by linearly combining four entries of $\mathbf{A}$. Give the concrete formulas in the form

$$(\tilde{\mathbf{A}})_{ij} = \underbrace{? \cdot (\mathbf{A})_{?,?} + ? \cdot (\mathbf{A})_{?,?} + ? \cdot (\mathbf{A})_{?,?} + ? \cdot (\mathbf{A})_{?,?}}\,,$$

filling in the question marks.

Four terms, because every $\tilde{b}_h^j$ is a linear comb. of two old b.f.

$(\tilde{A})_{ij} = a(\tilde{b}_h^j, \tilde{b}_h^i) =$ [use (2.2.2)]

$$1 \le i,j \le M: \quad (\tilde{\mathbf{A}})_{ij} = \mathsf{a}(b_h^{2j-1} + b_h^{2j}, b_h^{2i-1} + b_h^{2i}),$$

$$M+1 \le i,j \le N: \quad (\tilde{\mathbf{A}})_{ij} = \mathsf{a}(b_h^{2(j-M)-1} - b_h^{2(j-M)}, b_h^{2(i-M)-1} - b_h^{2(i-M)}),$$

$$M+1 \le i \le N, 1 \le j \le M: \quad (\tilde{\mathbf{A}})_{ij} = \mathsf{a}(b_h^{2j-1} + b_h^{2j}, b_h^{2(i-M)-1} - b_h^{2(i-M)}),$$

$$1 \le i \le M, M+1 \le j \le N: \quad (\tilde{\mathbf{A}})_{ij} = \mathsf{a}(b_h^{2(j-M)-1} - b_h^{2(j-M)}, b_h^{2i-1} + b_h^{2i}).$$

Then use bi-linearity of $a(\cdot,\cdot)$ & def. of $(A)_{k,l}$

$$1 \le i,j \le M: \quad (\tilde{\mathbf{A}})_{ij} = (\mathbf{A})_{2i,2j} + (\mathbf{A})_{2i,2j-1} + (\mathbf{A})_{2i-1,2j} + (\mathbf{A})_{2i-1,2j-1},$$

$$M+1 \le i,j \le N: \quad (\tilde{\mathbf{A}})_{ij} = (\mathbf{A})_{2(i-M)-1,2(j-M)-1} - (\mathbf{A})_{2(i-M),2(j-M)-1} - (\mathbf{A})_{2(i-M)-1,2(j-M)} + (\mathbf{A})_{2(i-M),2(j-M)},$$

$$M+1 \le i \le N, 1 \le j \le M: \quad (\tilde{\mathbf{A}})_{ij} = (\mathbf{A})_{2(i-N)-1,2j-1} - (\mathbf{A})_{2(i-M),2j-1} + (\mathbf{A})_{2(i-M)-1,2j} - (\mathbf{A})_{2(i-M),2j},$$

$$1 \le i \le M, M+1 \le j \le N: \quad (\tilde{\mathbf{A}})_{ij} = (\mathbf{A})_{2i-1,2(j-M)-1} + (\mathbf{A})_{2i,2(j-M)-1} - (\mathbf{A})_{2i-1,2(j-M)} - (\mathbf{A})_{2i,2(j-M)}$$

(2.2.4)

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③ d) To which $(\tilde{A})_{i,j}$ does $(A)_{k,\ell}$ contribute ?

Hint: Distinguish cases according to parity of (k,ℓ)

(2.2.4) $\Rightarrow$ $(A)_{\underbrace{2(i-M)}_{k},\underbrace{2j}_{\ell}} \longrightarrow -(\tilde{A})_{ij}$

[two even indices k,ℓ] $\rightarrow$ $i = k/2+M,\ j=\ell/2$

$\Downarrow$

will contribute to four entries of $\tilde{A}$

$$k,\ell \text{ even: } (\mathbf{A})_{k,\ell} \to (\tilde{\mathbf{A}})_{\frac{k}{2},\frac{\ell}{2}},\ (\tilde{\mathbf{A}})_{\frac{k}{2}+M,\frac{\ell}{2}+M},\ -(\tilde{\mathbf{A}})_{\frac{k}{2}+M,\frac{\ell}{2}},\ -(\tilde{\mathbf{A}})_{\frac{k}{2},\frac{\ell}{2}+M}, \tag{2.2.5}$$

$$k,\ell \text{ odd: } (\mathbf{A})_{k,\ell} \to (\tilde{\mathbf{A}})_{\frac{k+1}{2},\frac{\ell+1}{2}},\ (\tilde{\mathbf{A}})_{\frac{k+1}{2}+M,\frac{\ell+1}{2}+M},\ (\tilde{\mathbf{A}})_{\frac{k+1}{2},\frac{\ell+1}{2}+M},\ (\tilde{\mathbf{A}})_{\frac{k+1}{2}+M,\frac{\ell+1}{2}}, \tag{2.2.6}$$

$$k \text{ even, } \ell \text{ odd: } (\mathbf{A})_{k,\ell} \to (\tilde{\mathbf{A}})_{\frac{k}{2},\frac{\ell+1}{2}},\ (\tilde{\mathbf{A}})_{\frac{k}{2},\frac{\ell+1}{2}+M},\ -(\tilde{\mathbf{A}})_{\frac{k}{2}+M,\frac{\ell+1}{2}+M},\ -(\tilde{\mathbf{A}})_{\frac{k}{2}+M,\frac{\ell+1}{2}}, \tag{2.2.7}$$

$$k \text{ odd, } \ell \text{ even: } (\mathbf{A})_{k,\ell} \to (\tilde{\mathbf{A}})_{\frac{k+1}{2},\frac{\ell}{2}},\ -(\tilde{\mathbf{A}})_{\frac{k+1}{2}+M,\frac{\ell}{2}+M},\ (\tilde{\mathbf{A}})_{\frac{k+1}{2}+M,\frac{\ell}{2}},\ -(\tilde{\mathbf{A}})_{\frac{k+1}{2},\frac{\ell}{2}+M}. \tag{2.2.8}$$

e) Use triplet/COO format for sparse matrices

$$[(i_\ell, j_\ell, a_\ell)]_{\ell=1}^{L},\quad L\in\mathbb{N},\quad i_\ell\in\{1,\dots,n\},\quad j_\ell\in\{1,\dots,m\},\quad a_\ell\in\mathbb{R},$$

$$(\mathbf{M})_{k,q} = \sum_{\ell:\ i_\ell=k,\ j_\ell=q} a_\ell,\quad k\in\{1,\dots,n\},\quad q\in\{1,\dots,m\}.$$

Implement a C++ function

```
std::vector<Eigen::Triplet<double>> transformCOOmatrix(
   const std::vector<Eigen::Triplet<double>> &A);
```

that takes the Galerkin matrix $\mathbf{A}$ in triplet format and returns $\tilde{\mathbf{A}}$ als in triplet format.
It can be taken for granted that the argument matrix has no rows or columns with all zero entries.

Algorithm: Foreach (triplet (I,J,a) of A)
[According to parity of (I,J)
→ Add four triplets for $\tilde{A}$ with value $\pm a$]

f)

Theorem 2.2.10. Identity Galerkin matrices

Consider the Galerkin discretization of a symmetric positive-definite [Lecture → Def. 0.2.1.21] bilinear form $\mathsf{a}(\cdot,\cdot)$ using a finite-dimensional space $V_{0,h}$. Then there is a basis $\mathfrak{B}_h$ of $V_{0,h}$ such that the associated Galerkin matrix is the indentity matrix.

Hint: Diagonalization

$A \mathrel{\hat=}$ Galerkin matrix w.r.t. basis $\mathfrak{B}_h$ of $V_{0,h}$

$a(\cdot,\cdot)$ s.p.d. $\Rightarrow$ A s.p.d.

LA Thm: $\exists\, Q\in\mathbb{R}^{N,N}, D\in\mathbb{R}^{N,N}$ diagonal:

$N := \dim V_{0,h}$

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$$Q^T A Q = D \quad , \quad Q^T Q = I$$

A s.p.d. $\Rightarrow$ D has positive diagonal

$\triangleright$ $\underbrace{(QD^{-1/2})^T}_{=: S^T \text{ regular}} A \, (QD^{-1/2}) = I$

$=: S^T$ regular : gives change of basis matrix !

$\square$

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