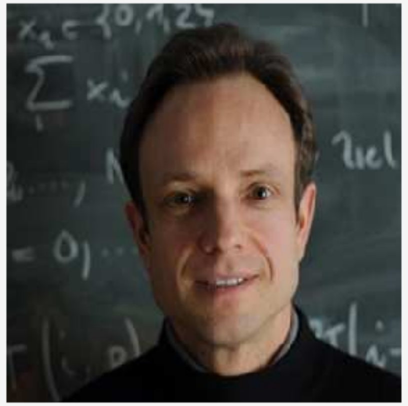
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Homework Problem Video Tutorial

Problem 1-5: A Second-Order Elliptic Transmission Problem



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$$u \in V_0: \quad \underbrace{\int_0^1 \frac{du}{dx}\frac{dv}{dx}\,\mathrm{d}x + u(1)v(1)}_{=:\mathrm{a}(u,v)} = \underbrace{v(1/2)}_{=:\ell(v)} \quad \forall v \in V_0\,, \tag{1.5.1}$$

↳ point evaluation functional

$V_0 = \{ v \in H^1(]0,1[) : v(0) = 0 \}$

a) Continuity of $a(\cdot,\cdot)$

To show: $|a(u,v)| \le C \|u\|_1 \|v\|_1 \quad \forall u,v \in V_0$

HINT 2 for (1-5.a): You may cite the result of Problem 1-3 that

$$|v(x)| \le |v|_{H^1(]0,1[)} \quad \forall x \in [0,1]\,, \quad \forall v \in H^1_0(]0,1[) / V_0 \tag{1.5.4}$$

which can easily be generalized to V_0. This estimate can also be deduced from the multiplicative trace inequality of [Lecture → Thm. 1.9.0.10] for $d = 1$.

$$a(u,v) \le \underbrace{\left\|\frac{du}{dx}\right\|_0}_{\le \|u\|_1} \underbrace{\left\|\frac{dv}{dx}\right\|_0}_{\le \|v\|_1} + \underbrace{|u(1)|}_{\le \|u\|_1}\,\underbrace{|v(1)|}_{\le \|v\|_1}$$

Cauchy-Schwarz inequality in $L^2(]0,1[)$

$$\le 2 \|u\|_1 \|v\|_1$$

b) $a(v,v) = |v|_1^2 + v(1)^2 \ge 0$

$a(v,v) = 0 \Rightarrow \|v'\|_0 = 0 \Rightarrow v' = 0$

$\Rightarrow v \equiv \text{const}$ & $v(1) = 0 \Rightarrow v \equiv 0$

$\Rightarrow a(\cdot,\cdot)$ s.p.d.

c) Continuity of ℓ w.r.t. $\|\cdot\|_a$?

$|\ell(v)| = |v(1/2)| \le |v|_1 \le \|v\|_a \quad \forall v \in V_0$

② $u \in V_0$: $\underbrace{\int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx + u(1)v(1)}_{=:a(u,v)} = \underbrace{v(1/2)}_{=:\ell(v)} \quad \forall v \in V_0$, (1.5.1)

d) ⚠ Derivation of strong form of BVP requires smoothness, which is not available for (1.5.1), because the source function $f = \delta(x-\frac{1}{2})$ is not smooth.

Idea: Apply IBP on sub-intervals $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$

$-\int_0^1 u'' v\,dx = 0 \quad \triangleright \quad u'' = 0$ in $]0,\frac{1}{2}[$

for test functions v compactly supported in $]0,\frac{1}{2}[$

e) BDC in $x = 1$?

Idea: Test functions v that vanish on $[0,\frac{1}{2}]$ + IBP

$$-\int_{1/2}^1 \cancel{\frac{d^2u}{dx^2}(x)}\, v(x)\,dx + \frac{du}{dx}(1)v(1) + u(1)v(1) = 0 \quad \forall v \in C^\infty([1/2,1]),\ v(1/2) = 0\,.$$

↳ Use ODE $u'' = 0$

$$\Downarrow \quad \underbrace{u'(1) + u(1) = 0}_{\text{BDC at } x=1}$$

f) Transmission conditions at $x = \frac{1}{2}$

(I) "built-into" $V_0 \subset H^1(]0,1[)$:

$\Rightarrow$ u continuous at $x = \frac{1}{2}$

(II) t.c. built into the variational problem:

Use $v \in C_0^\infty(]0,1[)$ & IBP

$$-\int_{[0,1]\setminus\{1/2\}} \underbrace{\cancel{\frac{d^2u}{dx^2}(x)}}_{=0 \text{ by (ODE)}} v(x)\,dx + \frac{du}{dx}(1/2-)v(1/2) - \frac{du}{dx}(1/2+)v(1/2) = v(1/2) \quad \forall v \in C_0^\infty(]0,1[)\,,$$

$$\Downarrow$$

$$\underbrace{\frac{du}{dx}(1/2-)}_{\text{"from left"}} - \underbrace{\frac{du}{dx}(1/2+)}_{\text{"from right"}} = 1\,. \quad (1.5.9)$$

$\triangleright$ Jump of of heigh 1 of u'

g) Solution of (1.5.1)

$u'' = 0$ in $x \neq \frac{1}{2}$, $u(0) = 0$ $\Rightarrow$ $u(x) = \begin{cases} \alpha x & , x \in [0,\frac{1}{2}[\\ \beta x + \gamma & , x \in]\frac{1}{2},1[\end{cases}$

$$\left.\begin{array}{lll} \text{Cont. @ } x = \frac{1}{2}: & \frac{1}{2}\alpha - \frac{1}{2}\beta - \gamma & = 0 \\ \text{T.C. II}: & \alpha - \beta & = 1 \\ & 2\beta + \gamma & = 0 \end{array}\right\} \quad 3\times 3 \text{ LSE} \quad \begin{array}{l} \alpha = 3/4 \\ \beta = -1/4 \\ \gamma = 1/2 \end{array}$$