

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

Prof. R. Hiptmair, SAM, ETH Zurich

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(C) Seminar für Angewandte Mathematik, ETH Zürich

[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 2-7: Properties of Householder reflections

[Lecture → Section 3.3.3] describes an orthogonal transformation that can be used to map a vector onto another vector of the same length: the Householder transformation [Lecture → Eq. (3.3.3.12)]. In this problem we examine properties of the “Householder matrices” and construct some of them.

Simple implementation in C++/EIGEN is requested

Householder transformation are described by square matrices of the form

$$\mathbf{H} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^H \quad \text{with} \quad \mathbf{v}^H\mathbf{v} = 1, \quad \mathbf{v} \in \mathbb{C}^n. \quad (2.7.1)$$

We study a few important properties of this class of matrices and delve into the geometric meaning of Householder reflections.

(2-7.a) ☐ Prove the following properties:

- i) $\mathbf{H}\mathbf{H} = \mathbf{I}$.
- ii) $\mathbf{H}\mathbf{H}^H = \mathbf{I}$.
- iii) $|\det(\mathbf{H})| = 1$.
- iv) For every $\mathbf{s} \in \mathbb{C}^n$ perpendicular to $\mathbf{v}$ (i.e. $\mathbf{v}^H\mathbf{s} = 0$): $\mathbf{H}\mathbf{s} = \mathbf{s}$.
- v) For every $\mathbf{z} \in \mathbb{C}^n$ collinear to $\mathbf{v}$ (i.e. $\mathbf{z} = c\mathbf{v}$): $\mathbf{H}\mathbf{z} = -\mathbf{z}$.

SOLUTION of (2-7.a):

- i) $\mathbf{H}\mathbf{H} = (\mathbf{I} - 2\mathbf{v}\mathbf{v}^H)^2 = \mathbf{I} - 4\mathbf{v}\mathbf{v}^H + 4(\mathbf{v}\mathbf{v}^H)(\mathbf{v}\mathbf{v}^H) \stackrel{\mathbf{v}^H\mathbf{v}=1}{=} \mathbf{I}$.
- ii) $\mathbf{H}^H = (\mathbf{I} - 2\mathbf{v}\mathbf{v}^H)^H = \mathbf{I} - 2\mathbf{v}\mathbf{v}^H = \mathbf{H}, \implies \mathbf{H}\mathbf{H}^H = \mathbf{H}\mathbf{H} \stackrel{i)}{=} \mathbf{I}$.
- iii) $\mathbf{H}\mathbf{H}^H = \mathbf{I} \implies \mathbf{H}^H = \mathbf{H}^{-1} \implies \mathbf{H} \text{ is unitary}$
 $\implies \forall \lambda_i \text{ eigenvalue of } \mathbf{H}, |\lambda_i| = 1 \implies |\det \mathbf{H}| = \left| \prod_{i=1}^n \lambda_i \right| = 1$.
- iv) $\mathbf{H}\mathbf{s} = \mathbf{s} - 2\mathbf{v}\mathbf{v}^H\mathbf{s} \stackrel{\mathbf{v}^H\mathbf{s}=0}{=} \mathbf{s}$.
- v) $\mathbf{H}\mathbf{z} = c\mathbf{H}\mathbf{v} = c(\mathbf{v} - 2\mathbf{v}\mathbf{v}^H\mathbf{v}) \stackrel{\mathbf{v}^H\mathbf{v}=1}{=} -c\mathbf{v} = -\mathbf{z}$.



(2-7.b) ☐

In real space the transformation $\mathbf{H}$ can be understood as a reflection across a hyperplane perpendicular to $\mathbf{v}$. Compute the matrix $\mathbf{H}$ and sketch the corresponding transformation for $\mathbf{v} = \frac{1}{\sqrt{10}}[1, 3]^T$, $n = 2$. What is the line of reflection?

SOLUTION of (2-7.b):

The matrix $\mathbf{H}$ is the following:

$$\mathbf{H} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}.$$

This transformation is a reflection across the “hyperplane” (= line, in $\mathbb{R}^2$) orthogonal to $\mathbf{v}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = x + 3y = 0.$$

The line of the reflection is defined through $x + 3y = 0$.



(2-7.c)

The matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

should be transformed to an upper triangular matrix using the Householder transformation ($\mathbf{H}\mathbf{C} = \mathbf{R}$).

Compute:

- the vector $\mathbf{v}$ that defines $\mathbf{H}$ (is it unique?);
- the matrix $\mathbf{H}$;
- the images of the columns of $\mathbf{C}$ under the transformation $\mathbf{H}$.

SOLUTION of (2-7.c):

We call $\mathbf{c}_1$ and $\mathbf{c}_2$ the columns of $\mathbf{C}$. Since the reflection preserves the norm of vectors, and since the image of the first column has to be zero in the second component, there are two possibilities for the reflection $\mathbf{H}$:

$$\mathbf{H}_1 : \mathbf{c}_1 \mapsto \begin{bmatrix} \|\mathbf{c}_1\| \\ 0 \end{bmatrix}, \quad \mathbf{H}_2 : \mathbf{c}_1 \mapsto \begin{bmatrix} -\|\mathbf{c}_1\| \\ 0 \end{bmatrix}.$$

Define $\mathbf{v}_1$ resp. $\mathbf{v}_2$ the vectors defining the transformation $\mathbf{H}_1$ resp. $\mathbf{H}_2$. Since $\|\mathbf{c}_1\| = \sqrt{5}$ we have, for $\mathbf{H}_1$:

$$\mathbf{v}_1 = \frac{\mathbf{c}_1 - \|\mathbf{c}_1\|\mathbf{e}_1}{\|\mathbf{c}_1 - \|\mathbf{c}_1\|\mathbf{e}_1\|} = \frac{1}{\sqrt{10 - 2\sqrt{5}}} \begin{bmatrix} 1 - \sqrt{5} \\ 2 \end{bmatrix},$$

therefore

$$\mathbf{H}_1 = \mathbf{I} - 2\mathbf{v}_1\mathbf{v}_1^H = \mathbf{I} - \frac{2}{5 - \sqrt{5}} \begin{bmatrix} 3 - \sqrt{5} & 1 - \sqrt{5} \\ 1 - \sqrt{5} & 2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

And for $\mathbf{H}_2$:

$$\mathbf{v}_2 = \frac{\mathbf{c}_1 + \|\mathbf{c}_1\|\mathbf{e}_1}{\|\mathbf{c}_1 + \|\mathbf{c}_1\|\mathbf{e}_1\|} = \frac{1}{\sqrt{10 + 2\sqrt{5}}} \begin{bmatrix} 1 + \sqrt{5} \\ 2 \end{bmatrix},$$

therefore

$$\mathbf{H}_2 = \mathbf{I} - 2\mathbf{v}_2\mathbf{v}_2^H = \mathbf{I} - \frac{2}{5 + \sqrt{5}} \begin{bmatrix} 3 + \sqrt{5} & 1 + \sqrt{5} \\ 1 + \sqrt{5} & 2 \end{bmatrix} = -\mathbf{H}_1.$$

The column $\mathbf{c}_2$ is mapped to

$$\mathbf{H}_1\mathbf{c}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\sqrt{5} \\ \sqrt{5} \end{bmatrix}.$$

Finally, the matrix $\mathbf{R}_1$ is

$$\mathbf{R}_1 = \mathbf{H}_1 \mathbf{C} = \mathbf{H}_1 [\mathbf{c}_1, \mathbf{c}_2] = \begin{bmatrix} \sqrt{5} & 2\sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}.$$



(2-7.d) Implement a function

```
void applyHouseholder(VectorXd &x, const VectorXd &v)
```

that efficiently computes $\mathbf{x} \leftarrow \mathbf{H}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$, for the Householder matrix ($\mathbf{v} \in \mathbb{R}^n$)

$$\mathbf{H} = \mathbf{I} - 2\mathbf{w}\mathbf{w}^\top, \quad \mathbf{w} := \frac{\mathbf{v}}{\|\mathbf{v}\|_2}.$$

What is the asymptotic complexity of your implementation for $n \rightarrow \infty$?

SOLUTION of (2-7.d):



(2-7.e) As we learned in [Lecture $\rightarrow$ Rem. 3.3.3.21], the sequence of Householder transformations $\mathbf{H}_1, \dots, \mathbf{H}_{n-1}$ effecting the conversion of an $n \times n$ matrix into upper triangular form ($\rightarrow$ [Lecture $\rightarrow$ § 3.3.3.11]) is represented by the sequence of their defining *unit vectors* $\mathbf{v}_i \in \mathbb{R}^n$, also following the convention that $(\mathbf{v}_i)_i \geq 0$: $\mathbf{H}_i := \mathbf{I} - 2\mathbf{v}_i\mathbf{v}_i^\top$. The vectors $\mathbf{v}_i$ are stored in the *strictly lower triangular* part of another $n \times n$ matrix $\mathbf{V} \in \mathbb{R}^{n,n}$. More precisely, we have

$$\mathbf{v}_i = [0, \dots, 0, (\mathbf{v}_i)_i, (\mathbf{V})_{i+1:n,i}]^\top \in \mathbb{R}^n, \quad i = 1, \dots, n-1.$$

Write a C++/EIGEN function

```
template <typename Scalar>
void applyHouseholder(VectorXd Xd &x,
                      const MatrixBase<Scalar> &V)
```

that computes $\mathbf{x} \leftarrow \mathbf{H}_{n-1}^{-1} \dots \mathbf{H}_1^{-1} \mathbf{x}$ based on Householder vectors stored in $\mathbf{V}$ as described above.

HINT 1 for (2-7.e): Instead of **MatrixXd** we specify an abstract matrix type for the matrix argument, which also allows to pass matrices of other double-compatible scalar type and expressions derived from them. ┘

HINT 2 for (2-7.e): The vector entry $(\mathbf{v}_i)_i$ can be computed, since we know $(\mathbf{v}_i)_i \geq 0$ and $\|\mathbf{v}_i\|_2 = 1$. ┘

SOLUTION of (2-7.e):



End Problem 2-7