

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

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Autumn Term 2021, Version of October 2, 2022
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[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 3-5: Conditional minimum with SVD

A *Singular Value Decomposition* (SVD, see [Lecture → Def. 3.4.1.3]) is the generalized version of the diagonalization process for any $m \times n$ matrix. We will see that the diagonal and orthogonal matrices arising from an SVD have certain properties such that it is immediate to identify the minimum of a least squares functional for that matrix. In other words, computing the SVD of a matrix is equivalent to solving its corresponding least square problem.

The solution of this problem is discussed in [Lecture → Section 3.4.4.1]; the algorithm is implemented in [Lecture → Code 3.4.4.2].

Let $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ be the singular value decomposition of $\mathbf{A} \in \mathbb{C}^{m,n}$, $m \geq n$. Consider the following problem:

$$\mathbf{x}^* = \underset{\|\mathbf{x}\|_2=1}{\operatorname{argmin}} \{ \|\mathbf{A}\mathbf{x}\|_2 \} \quad (3.5.1)$$

(3-5.a) ☞ (45 min.) Show that the solution of (3.5.1) is:

$$\mathbf{x}^* = \mathbf{v}_n := \mathbf{V}_{:,n} \quad (3.5.2)$$

Show that the following holds:

$$\|\mathbf{A}\mathbf{v}_n\|_2 = \sigma_n := \Sigma_{n,n}. \quad (3.5.3)$$

HINT 1 for (3-5.a): For your proof, exploit the properties of orthogonal matrices and define $\mathbf{y} := \mathbf{V}^\top \mathbf{x}$. Also remember that, by convention, the diagonal matrix $\mathbf{\Sigma}$ of a singular value decomposition presents the diagonal entries sorted in decreasing order: $(\Sigma)_{1,1} \geq (\Sigma)_{2,2} \cdots \geq (\Sigma)_{n,n}$. ┘

SOLUTION of (3-5.a):

See also [Lecture → Section 3.4.4.1]. For any orthogonal matrix $\mathbf{V}$ and any vector $\mathbf{x}$, the following holds:

$$\|\mathbf{V}\mathbf{x}\|^2 = (\mathbf{V}\mathbf{x})^\top \mathbf{V}\mathbf{x} = \mathbf{x}^\top \mathbf{V}^\top \mathbf{V}\mathbf{x} = \mathbf{x}^\top \mathbf{x} = \|\mathbf{x}\|^2. \quad (3.5.4)$$

Here, we used that $\mathbf{V}$ is orthogonal, which implies that $\mathbf{V}^\top \mathbf{V} = \mathbf{I}$. (The same holds for $\mathbf{V}^\top$ instead of $\mathbf{V}$).

Define $\mathbf{y} := \mathbf{V}^\top \mathbf{x}$. We have:

$$\|\mathbf{A}\mathbf{x}\| = \|\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \mathbf{x}\| = \|\mathbf{U}\mathbf{\Sigma}\mathbf{y}\| = \|\mathbf{\Sigma}\mathbf{y}\| \quad (3.5.5)$$

The first equality uses the singular value decomposition of $\mathbf{A}$. The second equality uses the definition $\mathbf{y} := \mathbf{V}^\top \mathbf{x}$. The last equality holds due to $\mathbf{U}$ being orthogonal.

Now, we show that:

$$\min_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|_2 = \sigma_n. \quad (3.5.6)$$

Firstly, we prove “ $\geq$ ”. We have the following:

$$\min_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| = \min_{\|\mathbf{y}\|=1} \|\mathbf{\Sigma}\mathbf{y}\| \quad (3.5.7)$$

$$= \min_{\|\mathbf{y}\|=1} \sqrt{\sigma_1^2 y_1^2 + \dots + \sigma_n^2 y_n^2} \quad (3.5.8)$$

$$\geq \min_{\|\mathbf{y}\|=1} \sqrt{\sigma_n^2 y_1^2 + \dots + \sigma_n^2 y_n^2} \quad (3.5.9)$$

$$= \sigma_n \min_{\|\mathbf{y}\|=1} \sqrt{y_1^2 + \dots + y_n^2} \quad (3.5.10)$$

$$= \sigma_n \min_{\|\mathbf{y}\|=1} \|\mathbf{y}\| = \sigma_n. \quad (3.5.11)$$

Here we first use the previously obtained equation and then compute $\Sigma \mathbf{y}$ using the fact that Σ is a diagonal matrix (when computing the norm).

Secondly, we prove “ $\leq$ ”. We can bound the minimum by:

$$\min_{\|\mathbf{x}\|=1} \|\mathbf{Ax}\| = \min_{\|\mathbf{y}\|=1} \|\Sigma \mathbf{y}\| \leq \|\Sigma \mathbf{e}_n\| = \sigma_n.$$

We have therefore proven the second equality (3.5.6), i.e. that $\min_{\|\mathbf{x}\|=1} \|\mathbf{Ax}\| = \sigma_n$, and $\mathbf{e}_n = \operatorname{argmin}_{\|\mathbf{y}\|=1} \|\Sigma \mathbf{y}\|$.

Let us reconstruct $\mathbf{x}$ from $\mathbf{y} = \mathbf{V}^\top \mathbf{x}$. Since $\mathbf{V}$ is orthogonal, we have that:

$$\mathbf{e}_n = \mathbf{y} = \mathbf{V}^\top \mathbf{x} \implies \mathbf{x} = \mathbf{V} \mathbf{y} = \mathbf{V} \mathbf{e}_n = \mathbf{V}_{:,n} =: \mathbf{v}_n. \quad (3.5.12)$$



End Problem 3-5, 45 min.