

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

Prof. R. Hiptmair, SAM, ETH Zurich

Autumn Term 2021, Version of October 2, 2022
(C) Seminar für Angewandte Mathematik, ETH Zürich

[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 5-2: Lagrange Polynomials

This short problem studies a particular aspect of the **Lagrange polynomials** introduced in [Lecture → § 5.2.2.3].

This exercise just requires basic knowledge of [Lecture → Section 5.2.2]

As in [Lecture → Eq. (5.2.2.4)] we denote by L_i the i -th Lagrange polynomial for given nodes $t_j \in \mathbb{R}$, $j = 0, \dots, n$, $t_i \neq t_j$, if $i \neq j$. Then the polynomial Lagrange interpolant p through the data points $(t_i, y_i)_{i=0}^n$ has the representation

$$p(x) = \sum_{i=0}^n y_i L_i(x) \quad \text{with} \quad L_i(x) := \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - t_j}{t_i - t_j}. \quad (5.2.1)$$

(5-2.a)  (10 min.) Show that

$$\sum_{i=0}^n L_i(t) = 1 \quad \forall t \in \mathbb{R}. \quad (5.2.2)$$

HINT 1 for (5-2.a): Which p will you get in (5.2.1), when setting $y_j = 1$? Recall the uniqueness of Lagrange interpolants asserted in [Lecture → Thm. 5.2.2.7]. 

SOLUTION of (5-2.a):

Consider the interpolation points $(t_i, 1)_{i=0}^n$. Clearly, an interpolating polynomial for these points is $p \equiv 1$. By uniqueness of Lagrange interpolants, we have $1 = p(x) = \sum_{i=0}^n L_i(x)$.



(5-2.b)  (15 min.) Show that

$$\sum_{i=0}^n L_i(0) t_i^j = \begin{cases} 1 & \text{for } j = 0, \\ 0 & \text{for } j = 1, \dots, n. \end{cases} \quad (5.2.3)$$

(Here, read t_i^j as t_i raised to the power j .)

HINT 1 for (5-2.b): Consider $y_i = t_i$ in (5.2.1) for $i = 0, \dots, n$. What will be the Lagrange interpolant p in this case? 

SOLUTION of (5-2.b):

For $j = 0$, see the previous problem. Consider $j > 0$ and the interpolation points $(t_i, t_i^j)_{i=0}^n$. Clearly, an interpolating polynomial for these points is $p(x) \equiv x^j$. By uniqueness of Lagrange interpolants (and $j \leq n$), we have $x^j = p(x) = \sum_{i=0}^n L_i(x) t_i^j$. Evaluating at $x = 0$: $0 = \sum_{i=0}^n L_i(0) t_i^j$.



(5-2.c) 🕒 (15 min.) Show that $p(x)$ can also be written as

$$p(x) = \omega(x) \sum_{i=0}^n \frac{y_i}{(x - t_i)\omega'(t_i)} \quad \text{with} \quad \omega(t) := \prod_{j=0}^n (t - t_j) . \quad (5.2.4)$$

Here ω' is the derivative of ω .

HINT 1 for (5-2.c): Apply the **product rule** for differentiation in a nested way to differentiate $\omega(t)$. ┘

SOLUTION of (5-2.c):

Simply exploiting the product rule for a product of several functions

$$\frac{d}{dx} \left\{ x \mapsto \prod_{j=1}^n f_j(x) \right\} = \sum_{i=1}^n f'_i(x) \prod_{j \neq i} f_j(x) , \quad (5.2.5)$$

we obtain:

$$\omega'(x) = \sum_{i=0}^n \prod_{\substack{j=0 \\ j \neq i}}^n (x - t_j) .$$

Since $(x - t_j) = 0$ when $x = t_j$, it follows that $\omega'(t_i) = \prod_{\substack{j=0 \\ j \neq i}}^n (t_i - t_j)$. Therefore:

$$\begin{aligned} \omega(x) \sum_{i=0}^n \frac{y_i}{(x - t_i)\omega'(t_i)} &= \sum_{i=0}^n \left[\frac{y_i}{(x - t_i) \prod_{\substack{j=0 \\ j \neq i}}^n (t_i - t_j)} \prod_{j=0}^n (x - t_j) \right] \\ &= \sum_{i=0}^n \left[\frac{y_i}{\prod_{\substack{j=0 \\ j \neq i}}^n (t_i - t_j)} \prod_{\substack{j=0 \\ j \neq i}}^n (x - t_j) \right] = \sum_{i=0}^n \left[y_i \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - t_j}{t_i - t_j} \right] = p(x) . \end{aligned}$$



End Problem 5-2 , 40 min.