

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

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[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 5-3: Generalized Lagrange polynomials for Hermite interpolation

[Lecture → Rem. 5.2.2.14] addresses a particular generalization of the Lagrange polynomial interpolation considered in [Lecture → Section 5.2.2]: **Hermite interpolation**, where beside the usual “nodal” interpolation conditions also the derivative of the interpolating polynomials is prescribed in the nodes, see [Lecture → Eq. (5.2.2.15)] (for $l_j = 1$ throughout). The associated generalized Lagrange polynomials were defined in [Lecture → Def. 5.2.2.17] and in this problem we aim to find concrete formulas for them in the spirit of [Lecture → Eq. (5.2.2.4)].

This is a purely theoretical problem.

Let $n + 1$ different nodes $t_i, i = 0, \dots, n, n \in \mathbb{N}$, be given. For numbers $y_j, c_j \in \mathbb{R}, j = 0, \dots, n$, Hermite interpolation seeks a polynomial $p \in \mathcal{P}_{2n+1}$ satisfying $p(t_i) = y_i$ and $p'(t_i) = c_i, i = 0, \dots, n$. Here p' designates the derivative of p .

The **generalized Lagrange polynomials** for Hermite interpolation $L_i, K_i \in \mathcal{P}_{2n+1}$ satisfy

$$L_i(t_j) = \delta_{ij}, \quad L'_i(t_j) = 0, \quad (5.3.1a)$$

$$K_i(t_j) = 0, \quad K'_i(t_j) = \delta_{ij}, \quad (5.3.1b)$$

for $i, j \in \{0, \dots, n\}$.

(5-3.a) ☑ (15 min.) Explicitly state the linear system of equations for the basis expansion coefficients of the Hermite interpolant with respect to the *monomial basis* $\{x \mapsto x^k, k = 0, \dots, 2n + 1\}$ of $\mathcal{P}_{2n+1}$.

HINT 1 for (5-3.a): Recall how the standard Lagrange polynomial interpolation is linked to the Vandermonde matrix presented in [Lecture → Rem. 5.2.2.10].

SOLUTION of (5-3.a):

Let $p(x) = \sum_{i=0}^{2n+1} a_i x^i$ be the interpolation polynomial in monomial form. We seek a linear system of equations $\mathbf{V}\mathbf{a} = \mathbf{b}$, for $\mathbf{a} = [a_0, \dots, a_{2n+1}]^\top \in \mathbb{R}^{2n+2}$. The first $n + 1$ equations of the system arise from the **interpolation conditions** $p(t_i) = y_i$:

$$y_j = p(t_j) = \sum_{i=0}^{2n+1} a_i t_j^i, \quad j = 0, \dots, n.$$

cf. the Vandermonde matrix. Differentiating the polynomial in monomial representation we get $p'(x) = \sum_{i=1}^{2n+1} i a_i x^{i-1}$. Using the **interpolation condition** $p'(t_i) = c_i$ for the derivative gives $n + 1$ more rows of the linear system:

$$c_j = p'(t_j) = \sum_{i=1}^{2n+1} a_i i t_j^{i-1}, \quad j = 0, \dots, n.$$

Hence, we find $\mathbf{b} = [y_0, \dots, y_n, c_0, \dots, c_n]^\top$ and the system matrix $\mathbf{V}$ turns out as

$$\mathbf{V} = \begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^{2n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{2n+1} \\ 0 & 1 & 2t_0 & \cdots & (2n+1)t_0^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2t_n & \cdots & (2n+1)t_n^{2n} \end{bmatrix} \in \mathbb{R}^{2n+2, 2n+2}.$$



(5-3.b) ☐ (15 min.) [depends on Sub-problem (5-3.a)]

Give concrete formulas for K_0, K_1, L_0, L_1 for $n = 1, t_0 = -1, t_1 = 1$.

HINT 1 for (5-3.b): Study [Lecture → Ex. 5.2.2.18] and consult [Lecture → Eq. (5.3.3.5)].



SOLUTION of (5-3.b):

We can find the generalized Lagrange polynomials in monomial form by solving the linear system derived in Sub-problem (5-3.a). In this special case we have

$$\mathbf{V} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \in \mathbb{R}^{4,4}.$$

By (5.3.1) the corresponding right-hand side vectors are just the Cartesian coordinate vectors:

$$L_0 \leftrightarrow \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad L_1 \leftrightarrow \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad K_0 \leftrightarrow \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad K_1 \leftrightarrow \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus we obtain the monomial expansion coefficients of L_0, L_1, K_0, K_1 as the columns of $\mathbf{V}^{-1}$:

$$\mathbf{V}^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 2 & 1 & -1 \\ -3 & 3 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

The resulting polynomials can also be written as

$$\begin{aligned} L_0(t) &= 3\left(\frac{1-t}{2}\right)^2 - 2\left(\frac{1-t}{2}\right)^3, & L_1(t) &= 3\left(\frac{1+t}{2}\right)^2 - 2\left(\frac{1+t}{2}\right)^3, \\ K_0(t) &= 2\left[\left(\frac{1-t}{2}\right)^2 - \left(\frac{1-t}{2}\right)^3\right], & K_1(t) &= -2\left[\left(\frac{1+t}{2}\right)^2 - \left(\frac{1+t}{2}\right)^3\right]. \end{aligned}$$

Note that $L_0(t) = L_1(-t)$, $K_0(t) = K_1(-t)$, which reflects the symmetry of the interval $[-1, 1]$.



(5-3.c) ☐ (10 min.) Find the general formula describing all polynomials in t

1. of degree 2 that have a double zero in $t = a, a \in \mathbb{R}$,
2. of degree $n > 2$ that have a double zero in $t = a, a \in \mathbb{R}$.

A polynomial p is said to have a **double zero** in $t = a$, if $p(a) = 0$ and $p'(a) = 0$.

SOLUTION of (5-3.c):

1. Any parabola that touches the x -axis in $x = a$ is of the form $x \mapsto c(x - a)^2$ with some $c \in \mathbb{R}$.

2. Every polynomial $p \in \mathcal{P}_n$, $n \geq 2$, with double zero in $t = a$ can be written as

$$p(t) = (t - a)^2 q(t) \quad \text{for some } q \in \mathcal{P}_{n-2}.$$



(5-3.d) ☐ (15 min.) Let $g_1, \dots, g_m$, $m \in \mathbb{N}$, be functions in $C^1(I)$ (space of continuously differentiable functions). Find a formula for the derivative of $h(t) = \prod_{k=1}^m g_k(t)$.

SOLUTION of (5-3.d):

Group the terms of the product according to

$$h(t) = g_1(t) \cdot (g_2(t) \cdots g_m(t)),$$

and apply the standard product rule $(fg)' = f'g + fg'$. Thus, by simple induction with respect to the length m of the product, one shows

$$h'(t) = \sum_{j=1}^m \left\{ g_j'(t) \cdot \prod_{\substack{k=1 \\ k \neq j}}^m g_k(t) \right\}. \quad (5.3.2)$$



(5-3.e) ☐ (30 min.) [depends on Sub-problem (5-3.c), Sub-problem (5-3.d)]

Find general formulas for the polynomials L_i, K_i as defined through (5.3.1). Of course, these formulas will involve the nodes t_i .

HINT 1 for (5-3.e): Of course, the insights obtained in Sub-problem (5-3.c) and Sub-problem (5-3.d) should be used. ┘

HINT 2 for (5-3.e): Note that the polynomials L_i and K_i have double zeros in t_j , $j \neq i$. According to Sub-problem (5-3.c) they can be written as

$$\begin{aligned} L_i(t) &= (\alpha_i t + \delta_i) \prod_{\substack{j=0 \\ j \neq i}}^n (t - t_j)^2, \\ K_i(t) &= (\gamma_i t + \beta_i) \prod_{\substack{j=0 \\ j \neq i}}^n (t - t_j)^2. \end{aligned} \quad (5.3.3)$$

Then determine α_i , δ_i , γ_i and β_i from (5.3.1) using the interpolation conditions and the result of Sub-problem (5-3.d). ┘

SOLUTION of (5-3.e):

To begin with we set

$$h_i(t) := \prod_{\substack{j=0, \dots, n \\ j \neq i}} (t - t_j)^2 \in \mathcal{P}_n, t \in \mathbb{R},$$

and apply (5.3.2): for $t \neq t_i, i = 0, \dots, n$,

$$h'_i(t) = 2 \sum_{j \neq i} (t - t_j) \prod_{k \neq i, j} (t - t_k)^2 = 2h_i(t) \cdot \sum_{j \neq i} \frac{1}{t - t_j} \quad \blacktriangleright \quad q_i(t) := \frac{h'_i(t)}{h_i(t)} = 2 \sum_{j \neq i} \frac{1}{t - t_j}.$$

The interpolation conditions (5.3.1) give us $L_i(t_i) = K'_i(t_i) = 1$ and $K_i(t_i) = L'_i(t_i) = 0$. Using the representations from (5.3.3) we find equations for α_i and δ_i

$$\begin{aligned} 1 &= L_i(t_i) = (\alpha_i t_i + \delta_i) h_i(t_i), \\ 0 &= L'_i(t_i) = \alpha_i h_i(t_i) + (\alpha_i t_i + \delta_i) h'_i(t_i), \end{aligned}$$

which leads to the linear system of equations:

$$\begin{bmatrix} t_i h_i(t_i) & h_i(t_i) \\ h_i(t_i) + t_i h'_i(t_i) & h'_i(t_i) \end{bmatrix} \begin{bmatrix} \alpha_i \\ \delta_i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We find the solution

$$\begin{bmatrix} \alpha_i \\ \delta_i \end{bmatrix} = \frac{1}{h_i(t_i)^2} \begin{bmatrix} -h'_i(t_i) \\ h_i(t_i) + t_i h'_i(t_i) \end{bmatrix} = \frac{1}{h_i(t_i)} \begin{bmatrix} -q_i(t_i) \\ 1 + t_i q_i(t_i) \end{bmatrix}.$$

This means

$$L_i(t) = \frac{1}{h_i(t_i)} (-q_i(t_i)t + 1 + t_i q_i(t_i)) h_i(t). \quad (5.3.4)$$

The coefficients β_i and γ_i for K_i can be computed in a similar way from

$$\begin{aligned} 0 &= K_i(t_i) = (\gamma_i t_i + \beta_i) h(t_i), \\ 1 &= K'_i(t_i) = \gamma_i h(t_i) + (\gamma_i t_i + \beta_i) h'_i(t_i), \end{aligned}$$

which can be recast as a linear system of equations:

$$\begin{bmatrix} t_i h_i(t_i) & h_i(t_i) \\ h_i(t_i) + t_i h'_i(t_i) & h'_i(t_i) \end{bmatrix} \begin{bmatrix} \gamma_i \\ \beta_i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We find the solution

$$\begin{bmatrix} \gamma_i \\ \beta_i \end{bmatrix} = -\frac{1}{h_i(t_i)^2} \begin{bmatrix} h'_i(t_i) & -h_i(t_i) \\ -h_i(t_i) - t_i h'_i(t_i) & t_i h_i(t_i) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{h_i(t_i)} \begin{bmatrix} 1 \\ -t_i \end{bmatrix}.$$

Written explicitly,

$$K_i(t) = \frac{1}{h_i(t_i)} (t - t_i) h_i(t).$$



End Problem 5-3, 85 min.