

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

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[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 5-7: Linear monotonicity-preserving C^1 -interpolation scheme

In Sections [Lecture → Section 5.3.2], [Lecture → Section 5.3.3.2], [Lecture → Section 5.3], different shape preserving operator are introduced. In the first case (piecewise linear functions) the interpolating function is continuous but not C^1 , in the two remaining cases (Hermite polynomials and splines) the interpolating operator is not linear. In all these examples the monotonicity is preserved, i.e., monotonic data yield monotonic interpolants. This problem shows that monotonicity preserving interpolators that are both C^1 and linear present an unpleasant and unexpected feature.

A purely theoretical problem. As preparation study [Lecture → § 5.1.0.21] and [Lecture → Def. 5.1.0.25] together with [Lecture → Section 5.3.3.2].

The goal of this problem is to prove the following theorem.

Theorem 5.7.1. Linear monotonicity preserving interpolation operator

Given the node set $\mathcal{T} = \{t_0, \dots, t_n\} \subset [t_0, t_n]$, with $t_0 < t_1 < \dots < t_n$, consider an interpolation operator $I_{\mathcal{T}} : \mathbb{R}^{n+1} \rightarrow C^0([t_0, t_n])$, for which, by definition, we have $(I_{\mathcal{T}}(\mathbf{y}))(t_j) = y_j$ for every $j = 0, \dots, n$ and every vector of data values $\mathbf{y} = [y_0, \dots, y_n]^T \in \mathbb{R}^{n+1}$. Assume that the operator satisfies

- i) smoothness: $I_{\mathcal{T}}(\mathbf{y})$ belongs to $C^1([t_0, t_n])$ for any $\mathbf{y} \in \mathbb{R}^{n+1}$,
- ii) linearity: $I_{\mathcal{T}}(\alpha \mathbf{y} + \beta \mathbf{z}) = \alpha I_{\mathcal{T}}(\mathbf{y}) + \beta I_{\mathcal{T}}(\mathbf{z})$, for every $\alpha, \beta \in \mathbb{R}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{n+1}$,
- iii) (positive) monotonicity preserving: if $y_j \leq y_{j+1}, \forall j = 0, \dots, n-1$, then $I_{\mathcal{T}}(\mathbf{y})$ is a monotonic non-decreasing function.

Then, for any $\mathbf{y} \in \mathbb{R}^{n+1}$, the derivative of $I_{\mathcal{T}}(\mathbf{y})$ in the interior nodes vanishes:

$$(I_{\mathcal{T}}(\mathbf{y}))'(t_j) = 0, \quad \forall j \in \{1, \dots, n-1\}, \quad \forall \mathbf{y} \in \mathbb{R}^{n+1}. \quad (5.7.2)$$

(5-7.a) 🕒 (15 min.)

Show that the assertion (5.7.2) holds for the special data vectors

$$\mathbf{y} = \mathbf{y}^{(j)} := [\underbrace{0, \dots, 0}_{j \text{ times}}, 1, \dots, 1]^T, \quad j = 0, \dots, n.$$

HINT 1 for (5-7.a): Recall the considerations in the beginning of [Lecture → Section 5.3.3.2].

SOLUTION of (5-7.a):

- The data $\mathbf{y}^{(j)}$ are increasing and, owing to monotonicity preservation, so must be the interpolant $I_{\mathcal{T}}(\mathbf{y}^{(j)})$.
- We have $(\mathbf{y}^{(j)})_1 = \dots = (\mathbf{y}^{(j)})_j = 0$ and $(\mathbf{y}^{(j)})_{j+1} = \dots = (\mathbf{y}^{(j)})_{n+1} = 1$. Thus, the function $I_{\mathcal{T}}(\mathbf{y}^{(j)})$ is constant in $[t_0, t_{j-1}]$ and in $[t_j, t_n]$. Since it is also C^1 -smooth, its derivative is identically zero in the two intervals (every time the interval itself does not degenerate into a point).
- All the points t_k , with $k = 1, \dots, n-1$, belong to one of these two intervals, therefore $(I_{\mathcal{T}}(\mathbf{y}^{(j)}))'(t_k) = 0$.

The derivative of $I_{\mathcal{T}}(\mathbf{y}^{(1)})$ is not necessarily zero in t_0 as well as the derivative of $I_{\mathcal{T}}(\mathbf{y}^{(n)})$ in t_n .



(5-7.b)  (10 min.) Prove that $\text{span}\{\mathbf{y}^{(j)}, j = 0, \dots, n\} = \mathbb{R}^{n+1}$.

SOLUTION of (5-7.b):

Every Cartesian coordinate vector of $\mathbb{R}^{n+1}$ can be written as a linear combination of the $\mathbf{y}^{(j)}$ s:

$$\begin{aligned} \mathbf{e}^{(j)} &:= (\underbrace{0, \dots, 0}_j, 1, 0, \dots, 0)^T = \mathbf{y}^{(j)} - \mathbf{y}^{(j+1)}, \quad \forall j = 0, \dots, n-1, \quad \mathbf{e}^{(n)} := (\underbrace{0, \dots, 0}_n, 1)^T = \mathbf{y}^{(n)}, \\ \Rightarrow \mathbb{R}^{n+1} &= \text{span}\{\mathbf{e}^{(j)}, j = 0, \dots, n\} \subseteq \text{span}\{\mathbf{y}^{(j)}, j = 0, \dots, n\} \subseteq \mathbb{R}^{n+1}, \\ &\Rightarrow \text{span}\{\mathbf{y}^{(j)}, j = 0, \dots, n\} = \mathbb{R}^{n+1}. \end{aligned}$$



(5-7.c)  (15 min.) [depends on Sub-problem (5-7.a) and Sub-problem (5-7.b)]

Invoke the insight gained in Sub-problem (5-7.a) to complete the proof of the theorem.

SOLUTION of (5-7.c):

Thanks to the result of Sub-problem (5-7.b), every $\mathbf{y} \in \mathbb{R}^{n+1}$ can be decomposed as $\mathbf{y} = \sum_{j=0}^n c_j \mathbf{y}^{(j)}$. Therefore, the derivatives in the nodes are

$$(I_{\mathcal{T}}(\mathbf{y}))'(t_k) \stackrel{\text{linearity of } I_{\mathcal{T}}}{=} \sum_{j=0}^n c_j (I_{\mathcal{T}}(\mathbf{y}^{(j)}))'(t_k) \stackrel{\text{Sub-problem (5-7.a)}}{=} 0, \quad k = 1, \dots, n-1.$$



End Problem 5-7, 40 min.