

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

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Autumn Term 2021, Version of October 2, 2022
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[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 5-9: Spaces of Piecewise Polynomial Functions

Piecewise polynomial functions, most prominently **splines**, are widely used for data interpolation and functions approximation. In this problem we examine some aspects of particular specimens.

A purely theoretical problem connected with [Lecture → Section 5.4].

For a mesh $\mathcal{M} := \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$ we consider the following three spaces of piecewise polynomials:

$$V_{-1} := \left\{ v : [a, b] \rightarrow \mathbb{R} : v|_{[t_{j-1}, t_j]} \in \mathcal{P}_2, j = 1, \dots, n \right\}, \quad (5.9.1)$$

$$V_0 := \left\{ v \in C^0([a, b]) : v|_{[t_{j-1}, t_j]} \in \mathcal{P}_2, j = 1, \dots, n \right\}, \quad (5.9.2)$$

$$V_1 := \left\{ v \in C^1([a, b]) : v|_{[t_{j-1}, t_j]} \in \mathcal{P}_2, j = 1, \dots, n \right\}, \quad (5.9.3)$$

$$V_2 := \left\{ v \in C^2([a, b]) : v|_{[t_{j-1}, t_j]} \in \mathcal{P}_2, j = 1, \dots, n \right\}. \quad (5.9.4)$$

Here, $\mathcal{P}_d$ stands for the space of polynomials $\mathbb{R} \rightarrow \mathbb{R}$ of degree $\leq d$.

(5-9.a) 🕒 (15 min.) Determine the dimensions of the four spaces defined above:

$$\dim V_{-1} = \boxed{},$$

$$\dim V_0 = \boxed{},$$

$$\dim V_1 = \boxed{},$$

$$\dim V_2 = \boxed{}.$$

SOLUTION of (5-9.a):

- The space V_{-1} has no continuity constraints at the nodes of the mesh. Thus, the dimensions of the local polynomial spaces on the mesh cells just add up:

$$\dim V_{-1} = n \cdot \dim \mathcal{P}_2 = 3n.$$

- Functions in the space V_1 have to satisfy a single continuity condition on every interface between two cells of the mesh, which means $n - 1$ constraints:

$$\blacktriangleright \dim V_0 = \dim V_{-1} - (n - 1) = 2n + 1.$$

- Another continuity condition for the first derivative is imposed in the case of the space V_1 . This introduces another $n - 1$ constraints.

$$\blacktriangleright \dim V_1 = \dim V_0 - (n - 1) = n + 2.$$

- Further $n - 1$ continuity conditions have to take into account in the case of V_2 :

$$\blacktriangleright \dim V_2 = \dim V_1 - (n - 1) = 3 = \dim \mathcal{P}_2.$$

In fact, we have $V_2 = \mathcal{P}_2$!



(5-9.b) (10 min.) Which of the spaces V_k , $k = -1, 0, 1$, coincides with one of the spline spaces $\mathcal{S}_{d,\mathcal{M}}$ introduced in [Lecture → Def. 5.4.1.1] in class?

$$V_{\boxed{}} = \mathcal{S}_{\boxed{},\mathcal{M}}.$$

SOLUTION of (5-9.b):

Spline spaces according to [Lecture → Def. 5.4.1.1] are piecewise polynomial spaces with maximal smoothness that do not collapse to spaces of simple polynomials:

► $V_1 = \mathcal{S}_{2,\mathcal{M}}.$



(5-9.c) (15 min.) Which of the following conditions for a function $v \in V_1$ are *sufficient* and which are *necessary* for v being non-decreasing, that is, for

$$a \leq x \leq y \leq b \quad \Rightarrow \quad v(x) \leq v(y) ?$$

(i) $v(t_{j-1}) \leq v(t_j)$ for all $j = 1, \dots, n$

☐ necessary

☐ sufficient

☐ neither

(ii) $v'(x) \geq 0$ for all $x \in [a, b]$

☐ necessary

☐ sufficient

☐ neither

(iii) $v'(\frac{1}{2}(t_{j-1} + t_j)) \geq 0$ for all $j \in \{1, \dots, n\}$

☐ necessary

☐ sufficient

☐ neither

(iv) $v'(t_j) \geq 0$ for all $j \in \{0, \dots, n\}$

☐ necessary

☐ sufficient

☐ neither

SOLUTION of (5-9.c):

- Obviously all the conditions are necessary, since $v \in C^1([a, b])$.
- (i) is not sufficient; $M = \{0, 1\}$ and $v(x) = x(x - 1)$ provides a counterexample
- (ii) is sufficient of course, because $v \in C^1([a, b])$. This is theorem from calculus.
- (iii) is not sufficient; $M = \{0, 1\}$ and $v(x) = x(2x - 1)$ provides a counterexample.

- (iv) is sufficient, because v' is continuous and piecewise linear. If such a function is ≥ 0 at all nodes, it will be ≥ 0 everywhere, which yields condition (ii).



In the sequel we consider the mesh (knot set) $\mathcal{M} = \{0, 1, 2\}$.

(5-9.d) (10 min.) Determine the parameters $a, b \in \mathbb{R}$ in the following function definition

$$v(x) := \begin{cases} ax + b & \text{for } 0 \leq x < 1, \\ 1 - x^2 & \text{for } 1 \leq x \leq 2, \end{cases}$$

such that $v \in V_1$.

$$a = \boxed{} \quad , \quad b = \boxed{} .$$

SOLUTION of (5-9.d):

- Since $v \in C^0([0, 2])$ (continuity at $x = 1$): $a \cdot 1 + b = 0$
- From $v \in C^1([0, 2])$: $a = -2$

Hence $a = -2$ and $b = 2$.



(5-9.e) (10 min.) Write down a $v \in V_1$ such that

$$v(0) = 0 \quad , \quad v(1) = 1 \quad , \quad v(2) = 0 .$$

$$v(x) = \begin{cases} \boxed{} & \text{for } 0 \leq x < 1, \\ \boxed{} & \text{for } 1 \leq x \leq 2. \end{cases}$$

SOLUTION of (5-9.e):

A possible choice is

$$v(x) = x(2 - x) .$$

In this case that function can be written in both boxes.

Note that this v is not the only solution. In fact, the space of possible solutions is one-dimensional: 3 conditions imposed on a functions from a 4-dimensional space.



End Problem 5-9 , 60 min.