

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

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[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 6-9: Finding domains of analyticity

The developments of [Lecture → Lemma 6.2.2.53], [Lecture → Rem. 6.2.3.26], and [Lecture → Section 6.5.3] make clear that knowledge about that maximal subdomain $D \subset \mathbb{C}$ of the complex plane to which an analytic function can be extended from its original real interval of definition is key to predicting the speed of exponential convergence of approximation schemes like Chebychev interpolation and trigonometric interpolation. In this problem we discuss the analytic extension of 1-periodic functions.

Depends on [Lecture → Rem. 6.2.2.67] and is related to [Lecture → § 6.5.3.16].

In order to analyze the trigonometric functions occurring in this problem, the following identities can be useful:

$$\begin{aligned}\sin(x + iy) &= \sin(x) \cosh(y) + i \cos(x) \sinh(y) \quad \forall x, y \in \mathbb{R}, \\ \cos(x + iy) &= \cos(x) \cosh(y) - i \sin(x) \sinh(y) \quad \forall x, y \in \mathbb{R}, \\ \sin^2(z) + \cos^2(z) &= 1 \quad \forall z \in \mathbb{Z}.\end{aligned}\tag{6.9.1}$$

The following result gives domains of analyticity of special functions:

Theorem 6.9.2. domains of analyticity of special functions

- Every polynomial is analytic on $\mathbb{C}$.
- Every rational function, that is, a quotient of two polynomials, is analytic in the complement of the set of zeros of its denominator.
- The functions $\exp$, $\sin$, and $\cos$ are analytic on $\mathbb{C}$.
- The functions $\log : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\sqrt{\cdot} : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ can be extended to analytic functions on $\mathbb{C} \setminus \mathbb{R}_0^-$.

(6-9.a) 🕒 (6 min.) We consider the 1-periodic function

$$f(t) = b \log(a + \sin(2\pi t)), \quad t \in \mathbb{R}, \quad b > 0, a > 1.\tag{6.9.3}$$

Determine the largest possible subset D of $\mathbb{C}$ to which f can be extended analytically.

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \operatorname{Re}(z) \in \boxed{}, \\ \operatorname{Im}(z) \in \boxed{} \end{array} \right\}.$$

SOLUTION of (6-9.a):

We write $P := \mathbb{C} \setminus D \subset \mathbb{C}$ for the complement of the domain of analyticity (the “domain of singularity”) of f . Multiplication with b does not have any impact on analyticity. By Thm. 6.9.2 we have

$$\begin{aligned}P &= \{z \in \mathbb{C} : a + \sin(2\pi z) \in \mathbb{R}_0^-\} \\ &= \{z \in \mathbb{C} : \operatorname{Im}(a + \sin(2\pi z)) = 0 \quad \wedge \quad \operatorname{Re}(a + \sin(2\pi z)) \leq 0\}.\end{aligned}$$

Next, we use (6.9.1) with $z = x + iy$, $x, y \in \mathbb{R}$:

$$\sin(2\pi z) = \sin(2\pi x) \cosh(2\pi y) + i \cos(2\pi x) \sinh(2\pi y),$$

(6-9.c) ☐ (6 min.)

We consider the 1-periodic function

$$f(t) = a\sqrt{b - \sin(2\pi t)}, \quad t \in \mathbb{R}, \quad b > 1, a > 1. \quad (6.9.5)$$

Determine the largest possible subset D of $\mathbb{C}$ to which f can be extended analytically.

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \operatorname{Re}(z) \in \boxed{} \\ \operatorname{Im}(z) \in \boxed{} \end{array} \right\}.$$

SOLUTION of (6-9.c):

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \operatorname{Re}(z) \in \mathbb{Z} + \frac{1}{4}, \\ \operatorname{Im}(z) \in \{y \in \mathbb{R} : |y| \geq q, \cosh(2\pi q) = b\} \end{array} \right\}.$$

▲

End Problem 6-9 , 18 min.