

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

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Autumn Term 2021, Version of October 2, 2022
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[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 7-2: Generalized “Hermite-type” quadrature formula

In this exercise we will meet a new class of quadrature formulas and then use one of them in an application.

This is a purely theoretical exercise. You may want to have a look at the methods used in [Lecture → Ex. 7.4.2.2], as they will come handy.

(7-2.a) ☞ Determine $A, B, C, x_1 \in \mathbb{R}$ such that the quadrature formula

$$\int_0^1 f(x) dx \approx Af(0) + Bf'(0) + Cf(x_1) \quad (7.2.1)$$

is exact for polynomials of highest possible degree.

SOLUTION of (7-2.a):

The quadrature is exact for every polynomial $p(x) \in \mathcal{P}^n$ if and only if it is exact for $1, x, x^2, \dots, x^n$. If we apply the quadrature to the first monomials:

$$\begin{aligned} 1 &= \int_0^1 1 dx = A \cdot 1 + B \cdot 0 + C \cdot 1 = A + C \\ \frac{1}{2} &= \int_0^1 x dx = A \cdot 0 + B \cdot 1 + C \cdot x_1 = B + Cx_1 \\ \frac{1}{3} &= \int_0^1 x^2 dx = A \cdot 0 + B \cdot 0 + C \cdot x_1^2 = Cx_1^2 \\ \frac{1}{4} &= \int_0^1 x^3 dx = A \cdot 0 + B \cdot 0 + C \cdot x_1^3 = Cx_1^3 \end{aligned}$$

Thus:

$$x_1 = \frac{3}{4}, C = \frac{16}{27}, B = \frac{1}{18}, A = \frac{11}{27}$$

On the other hand, we have that:

$$\frac{1}{5} = \int_0^1 x^4 dx \neq A \cdot 0 + B \cdot 0 + C \cdot x_1^4 = C \cdot x_1^4 = \frac{16}{27} \frac{81}{256}$$

Hence, the quadrature is exact for polynomials up to degree 3.



(7-2.b) ☞ Compute an approximation of $z(2)$ using the quadrature rule of Eq. (7.2.1), where the function z is defined as the solution of the initial value problem

$$z'(t) = \frac{t}{1+t^2}, \quad z(1) = 1. \quad (7.2.2)$$

HINT 1 for (7-2.b): Use the fundamental theorem of calculus and your result from the previous exercise.

SOLUTION of (7-2.b):

By the fundamental theorem of calculus we know that

$$z(2) - z(1) = \int_1^2 z'(x) dx, \quad (7.2.3)$$

hence, applying the quadrature rule and the transformation $x \mapsto x + 1$, we obtain:

$$z(2) = \int_0^1 z'(x+1) dx + z(1) \approx \frac{11}{27} \cdot z'(1) + \frac{1}{18} \cdot z''(1) + \frac{16}{27} \cdot z'\left(\frac{7}{4}\right) + z(1). \quad (7.2.4)$$

With $z''(x) = \frac{1-x^2}{(1+x^2)^2}$ and:

$$\begin{aligned} z(1) &= 1, \\ z'(1) &= \frac{1}{1+1^2} = \frac{1}{2}, \\ z''(1) &= \frac{1-1^2}{(1+1^2)^2} = 0, \\ z'\left(\frac{7}{4}\right) &= \frac{\frac{7}{4}}{1+\left(\frac{7}{4}\right)^2} = \frac{28}{65}, \end{aligned}$$

we obtain

$$z(2) = \int_0^1 z'(x+1) dx + z(1) \approx \frac{11}{27} \cdot \frac{1}{2} + \frac{1}{18} \cdot 0 + \frac{16}{27} \cdot \frac{28}{65} + 1 = 1.45897 \dots$$

For the sake of completeness, using the antiderivative of z' :

$$z(2) = \int_1^2 z'(x) dx + z(1) = \frac{1}{2} \log(x^2 + 1) \Big|_1^2 + 1 = 1.45815 \dots$$



End Problem 7-2