

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

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[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 7-7: Quadrature by transformation

In [Lecture → Rem. 7.4.3.10] a suitable transformation converted a singular integrand into a smooth one. In this problem we see another example. Also Problem 7-9 and Problem 7-1 cover this topic.

This problem does not contain an implementation part. Study [Lecture → Rem. 7.4.3.10] to learn about regularizing transformations.

For $f \in C^0([0, 2])$ we consider the definite improper integral with *singular integrand*

$$I(f) := \int_0^2 \sqrt{\frac{2-t}{t}} f(t) dt. \quad (7.7.1)$$

(7-7.a) ☞ Transform $I(f)$ into an integral over $[-1, 1]$, whose integrand is $\in C^\infty([-1, 1])$ provided that $f \in C^\infty([0, 2])$.

HINT 1 for (7-7.a): First split the integral into integrals over $[0, 1]$ and $[1, 2]$ in order to isolate the two different singularities of the integrand. Apply different transformations to the two parts. ┘

HINT 2 for (7-7.a): Use the substitution $s = \sqrt{t}$ to deal with the singularity in $t = 0$ and $s = \sqrt{2-t}$ to get rid of the singularity in $t = 2$. ┘

SOLUTION of (7-7.a):

We wish to transform the integral in $I(f)$ into an integral of a smooth function over a bounded interval. In order to this, write

$$I(f) = \int_0^1 \frac{\sqrt{2-t}}{\sqrt{t}} f(t) dt + \int_1^2 \frac{\sqrt{2-t}}{\sqrt{t}} f(t) dt.$$

Performing the substitutions $s = \sqrt{t}$ in the first integral and $s = \sqrt{2-t}$ in the second integral we obtain

$$I(f) = 2 \int_0^1 \sqrt{2-s^2} f(s^2) ds + 2 \int_0^1 \frac{s^2}{\sqrt{2-s^2}} f(2-s^2) ds = 2 \int_0^1 f_1(s) ds + 2 \int_0^1 f_2(s) ds$$

where $f_1(s) = \sqrt{2-s^2} f(s^2)$ and $f_2(s) = \frac{s^2}{\sqrt{2-s^2}} f(2-s^2)$. With the change of variables $t = 2s - 1$ we finally obtain

$$I(f) = \int_{-1}^1 f_1((t+1)/2) ds + \int_{-1}^1 f_2((t+1)/2) ds. \quad (7.7.2)$$

Observe that we never get close to the singularity of the root terms.



(7-7.b) ☞ (20 min.) For $n \in \mathbb{N}^*$, derive a family of $2n$ -point quadrature formulas

$$Q_n(f) = \sum_{j=1}^{2n} w_j^n f(c_j^n), \quad f \in C([0, 2])$$

for which $Q_n(f)$ converges to $I(f)$ exponentially as $n \rightarrow \infty$, if $f \in C^\infty([0, 2])$.

Remark. A more precise statement of the problem would be “, if f has an analytic extension to a complex neighborhood of $[0, 2]$ ”. This is, how the statement $f \in C^\infty([0, 2])$ above should be read.

SOLUTION of (7-7.b):

Applying Gauss quadrature to these integrals in (7.7.2), we define

$$Q_n(f) = \sum_{j=1}^n \tilde{w}_j^n f_1((\xi_j^n + 1)/2) + \sum_{j=1}^n \tilde{w}_j^n f_2((\xi_j^n + 1)/2),$$

where ξ_j^n and $\tilde{w}_j^n$ are nodes and weights of the Gauss quadrature of order n defined on $[-1, 1]$. (Alternatively, it is also possible to use the Clenshaw-Curtis quadrature rule.) By construction, we have that $Q_n(f) \rightarrow I(f)$ exponentially as $n \rightarrow \infty$ if f has an analytic extension to a neighborhood of $[1, 2]$, since this implies that the integrands have an analytic extension to a neighborhood of $[-1, 1]$.

Define now the nodes

$$c_j^n = \begin{cases} ((\xi_j^n + 1)/2)^2 & \text{if } j = 1, \dots, n, \\ 2 - ((\xi_{j-n}^n + 1)/2)^2 & \text{if } j = n + 1, \dots, 2n. \end{cases}$$

Therefore, the above expression can be written as

$$Q_n(f) = \sum_{j=1}^n \tilde{w}_j^n \sqrt{2 - c_j^n} f(c_j^n) + \sum_{j=1}^n \tilde{w}_j^n \frac{2 - c_{j+n}^n}{\sqrt{c_{j+n}^n}} f(c_{j+n}^n) = \sum_{j=1}^{2n} w_j^n f(c_j^n),$$

where the weights are given by

$$w_j^n = \begin{cases} \tilde{w}_j^n \sqrt{2 - c_j^n} & \text{if } j = 1, \dots, n, \\ \tilde{w}_{j-n}^n \frac{2 - c_j^n}{\sqrt{c_j^n}} & \text{if } j = n + 1, \dots, 2n. \end{cases}$$



(7-7.c) (10 min.) Given $n > 0$, determine the maximal $d \in \mathbb{N}$ such that $I(p) = Q_n(p)$ for every polynomial $p \in \mathcal{P}_{d-1}$.

SOLUTION of (7-7.c):

The above expression is never exact for polynomials f , because of the presence of the square roots. In other words, $d = 0$, for which $\mathcal{P}_{-1} = \emptyset$.



End Problem 7-7, 30 min.