

ETH Lectures 401-0663-00L Numerical Methods for Computer Science
401-2673-00L Numerical Methods for CSE

Homework Problems and Projects

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[Link](#) to the current version of this homework collection

0.1 General Information

0.1.1 Weekly Homework Assignments

All problems will be published in this single “.pdf” file. Every week, we publish a list of problem numbers on the lecture Moodle page. These are the current assignments for that week. Details are given on the lecture Moodle page.

0.1.2 Importance of Homework

Homework assignments are **not** mandatory. However, it is **very important** that you constantly exercise with the material you learn. “Solving” the homework assignments one week before the main exam by looking at the solutions will likely result in failure.

We provide hints and solutions from the beginning. It is your responsibility to look at the solutions only if you are stuck with a difficult problem and you tried to find a solution for a sufficient amount of time.

Most homework problems come with an estimate for the time required for its solution. These times are what we expect a bright and well-prepared student to require for that problem during an exam.

0.1.3 Corrections and Grading of Assignments

Homework assignments will not be graded.

However, you may submit your handwritten solutions by handing them to your tutor directly or uploading them via the Moodle upload interface, see the lecture Moodle page for instructions. The assistants will

Problem 8-3: Newton's method for $F(x) := \arctan x = 0$

The merely local convergence of Newton's method is notorious, see [Lecture → Section 8.5.2] and [Lecture → Ex. 8.5.4.1]. The failure of the convergence is often caused by the overshooting of Newton correction. In this problem we try to understand the observations made in [Lecture → Ex. 8.5.4.1].

Moderate implementation in C++ is requested.

(8-3.a) (20 min.) Find an equation satisfied by the smallest positive initial guess $x^{(0)}$ for which Newton's method does not converge when it is applied to $F(x) = \arctan x$.

HINT 1 for (8-3.a): Refer to [Lecture → Fig. 304] and find out for which initial guesses the Newton method oscillates between two values.

HINT 2 for (8-3.a):

Newton critical iteration

Clearly the solution is $x^* = 0$.

Graphical considerations may help you to find the solution, see Fig. 87. If $x^{(0)}$ is close to zero, the the Newton iteration will converge, for $|x^{(0)}|$ large it will diverge.

However, there are two initial guesses $\pm x^{(0)}$, for which $x^{(1)} = -x^{(0)}$ and those mark the boundary between the two sets.

The condition $x^{(1)} = -x^{(0)}$ gives you a non-linear equation that you can solve for $x^{(0)}$.

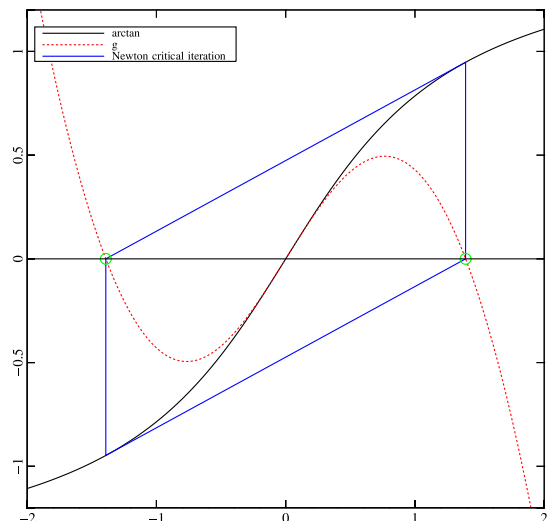


Fig. 87

SOLUTION of (8-3.a):

The function $\arctan(x)$ is **positive, increasing and concave** for positive x , therefore the first iterations of Newton's method with initial points $0 < x^{(0)} < y^{(0)}$ satisfy $y^{(1)} < x^{(1)} < 0$ (draw a sketch to see it). The function is **odd**, i.e., $\arctan(-x) = -\arctan(x)$ for every $x \in \mathbb{R}$, therefore the analogous holds for initial negative values ($y^{(0)} < x^{(0)} < 0$ gives $0 < x^{(1)} < y^{(1)}$). Moreover, opposite initial values give opposite iterations: if $y^{(0)} = -x^{(0)}$ then $y^{(n)} = -x^{(n)}$ for every $n \in \mathbb{N}$.

All these facts imply that, if $|x^{(1)}| < |x^{(0)}|$, then the absolute values of the following iterations will converge monotonically to zero. Vice versa, if $|x^{(1)}| > |x^{(0)}|$, then the absolute values of the Newton's iterations will diverge monotonically. Moreover, the iterations change sign at each step, i.e., $x^{(n+1)} < 0$.

It follows that the smallest positive initial guess $x^{(0)}$ for which Newton's method does not converge satisfies $x^{(1)} = -x^{(0)}$. This can be written as

$$x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})} = x^{(0)} - (1 + (x^{(0)})^2) \arctan x^{(0)} = -x^{(0)}.$$

Therefore, $x^{(0)}$ is a zero of the function

$$g(x) := 2x - (1 + x^2) \arctan x \quad \text{with} \quad g'(x) = 1 - 2x \arctan x.$$



(8-3.b) 🕒 (20 min.) Implement a C++ function

```
double newton_arctan(double x0_ = 2.0);
```

that uses Newton's method to find an approximation of such $x^{(0)}$. The argument `x0_` can be used to supply an initial guess for the iteration.

The considerations from Sub-problem (8-3.a) suggest that we use an initial guess in $[1, 2]$, we opt for $x^{(0)} = 2$.

SOLUTION of (8-3.b):

Newton's iteration to find the smallest positive initial guess reads

$$x^{(n+1)} = x^{(n)} - \frac{2x^{(n)} - (1 + (x^{(n)})^2) \arctan x^{(n)}}{1 - 2x^{(n)} \arctan x^{(n)}} = \frac{-x^{(n)} + (1 - (x^{(n)})^2) \arctan x^{(n)}}{1 - 2x^{(n)} \arctan x^{(n)}}.$$

C++-code 8.3.1: Solution of Sub-problem (8-3.b)

```
2  double newton_arctan(double x0_ = 2) {
3      // TO DO (9-3.c): define a suitable Newton iteration to find the
4      // critical value x0 for F(x) = arctan(x).
5      double x0 = x0_;
6      // START
7      double upd = 1;
8      double eps = std::numeric_limits<double>::epsilon();
9
10     while (upd > eps) {
11         // Newton iteration for g(x)
12         double x1 =
13             (-x0 + (1 - x0 * x0) * std::atan(x0)) / (1 - 2 * x0 * std::atan(x0));
14         // Relative size of the Newton update
15         upd = std::abs((x1 - x0) / x1);
16         x0 = x1;
17     }
18     // END
19     return x0;
20 }
```



End Problem 8-3, 40 min.