ETH Lecture 401-0663-00L Numerical Methods for CSE

End-term Exam

Autumn Term 2020

Dec 10, 2020, 10:15, Moodle online exam

Don't panic!

		%
Family Name		
First Name		
Department		
Legi Nr.		
Date	Dec 10, 2020	

Points:

	1	2	3	Total
max	6	8	6	20
achvd				

100% = 16 pts.

- This is an **open-book exam**.
- **Duration: 30 minutes.**
- The link to the Moodle exam is

  `https://moodle-app6.let.ethz.ch/mod/quiz/view.php?id=3140`
- Maximize your browser window and visit the exam page.
- Read and agree to the declaration of independence (Eigenständigkeitserklärung), and enter the password:

  NumCSE end-term exam password: `NumCSE2020`
- In case of technical problems, immediately contact `online-pruefungen@let.ethz.ch`.
- At 10:15 (CET), you may press "Start attempt" (Versuch beginnen). Once you do, a timer of 30 minutes is started.
- When the timer expires, your attempt is finished and submitted. You can no longer make changes to your answers.
- If you start your attempt later than 10:20, you will not be granted the full 30 minutes to finish your attempt.
- The exam comprises three problems. You can navigate between them by pressing "Next page" (Nächste Seite) and "Previous page" (Vorherige Seite).
- Once you are ready to submit the exam, press "Finish attempt" (Versuch beenden). You will be presented with a summary of your attempt. Press "Submit all and finish" (Abgabe).

- Some answers may only require numerical values (e.g. “2”, “-5”, “3.5”,...). Do not use other characters in these answers (e.g. “a=2”, “7/2”,...) and avoid whitespace.
- Some answers may require the use of mathematical notation. Please read the instructions below on how to effectively communicate mathematical expressions through plain text.
- Some questions may have multiple valid answers. Your answers will be graded by a human.
- You are advised to join the NumCSE Q&A ZOOM Meeting Room
  - Link: `https://ethz.zoom.us/j/198618336`
  - Meeting ID: **198 618 336**

  before logging in to the exam site. In case of problems or questions relating to the content of the exam, use the ZOOM chat to communicate them. The main examiner and head assistants will watch the chat activity.

When typing formulas, please adhere to the following conventions

- leave space between symbols
- `=`, `>`, `<`, `<=`, `>=` mean $=$, $>$, $<$, $\leq$, $\geq$,
- for brackets use `(`, `)`, `[`, `]`,
- elementary functions: `exp`, `sin`, `cos`, `log`, `sqrt`, `cosh`, `tan`, `sinh`, `arcsin`, `arccos`, `arctan`, `arsinh`
- exponentiation: `a^x` $\hat{=}$ a^x,
- fractions: `(1+x)/(1-x)` $\hat{=}$ $\dfrac{1+x}{1-x}$,
- sets `{ }`, `\in` $\hat{=}$ $\in$.
- set of numbers `R` $\hat{=}$ $\mathbb{R}$, `C` $\hat{=}$ $\mathbb{C}$, `Z` $\hat{=}$ $\mathbb{Z}$, `N` $\hat{=}$ $\mathbb{N}$, `N0` $\hat{=}$ $\mathbb{N}_0$,
- the number π: `pi` $\hat{=}$ π.
- Greek letters $\alpha, \beta, \gamma, \ldots$ $\hat{=}$ `alpha`, `beta`, `gamma`, ...

The exam employs the notations introduced in class, in particular [Lecture → Section 1.1.1]:

- $(\mathbf{A})_{i,j}$ to refer to the entry of the matrix $\mathbf{A} \in \mathbb{K}^{m,n}$ at position (i, j).
- $(\mathbf{A})_{:,i}$ to designate the ith-column of the matrix $\mathbf{A}$,
- $(\mathbf{A})_{i,:}$ to denote the i-th row of the matrix $\mathbf{A}$,
- $(\mathbf{A})_{i:j,k:\ell}$ to single out the sub-matrix $\left[(\mathbf{A})_{r,s}\right]_{\substack{i\leq r\leq j \\ k\leq s\leq \ell}}$ of the matrix $\mathbf{A}$,
- $(\mathbf{x})_k$ to reference the k-th entry of the vector $\mathbf{x}$,
- $\mathbf{e}_j \in \mathbb{R}^n$ to write the j-th Cartesian coordinate vector,
- $\mathbf{I}$ to denote the identity matrix,
- $\mathbf{O}$ to write a zero matrix,
- $\mathcal{P}_n$ for the space of (univariate polynomials of degree $\leq n$),
- and superscript indices in brackets to denote iterates: $\mathbf{x}^{(k)}$, etc.

By default, vectors are regarded as column vectors.

Problem 0-1: A cubic spline

Since they mimic the behavior of an elastic rod pinned at fixed points, see [Lecture → § 5.4.3.1], cubic splines are very popular for creating "aesthetically pleasing" interpolating functions. However, in this problem we look at a cubic spline from the perspective of its defining properties, see [Lecture → Def. 5.4.1.1], in order to become more familiar with the concept of spline function and the consequences of the smoothness required by the definition.

Related to [Lecture → Section 5.4.1]

(0-1.a) (6 pts.) For parameters $\alpha, \beta, \gamma, \delta, \sigma, \rho \in \mathbb{R}$ we define the function

$$s(t) = \begin{cases} -1 & \text{for} \quad -1 \leq t < 0\,, \\ \alpha t^3 + \beta t^2 + \gamma t + \delta & \text{for} \quad 0 \leq t < 1\,, \\ -t^3 + \sigma t^2 + \rho t + 2 & \text{for} \quad 1 \leq t \leq 2\,. \end{cases}$$

Determine the parameters $\alpha, \beta, \gamma, \delta, \sigma, \rho \in \mathbb{R}$ such that s is a cubic spline with respect to the knot set $\mathcal{M} := \{-1, 0, 1, 2\}$.

$$\alpha = \square\,, \quad \beta = \square \quad \gamma = \square\,,$$
$$\delta = \square\,, \quad \sigma = \square \quad \rho = \square\,.$$

SOLUTION of (0-1.a):

Since $s \in C^2([-1,2])$ is required for $s \in \mathcal{S}_{3,\mathcal{M}}$ [Lecture → Def. 5.4.1.1], at each node we have three continuity conditions for s, the first derivative s', and the second derivative s'':

- At $t_0 = 0$:

$$s(-1) = -1 \quad , \quad s'(-1) = 0 \quad , \quad s''(-1) = 0\,.$$
$$\blacktriangleright \quad s(t) = \alpha t^3 - 1 \quad \text{for} \quad 0 \leq t < 1\,.$$

  This means $\beta = \gamma = 0$ and $\delta = -1$.

- At $t = 1$

$$s|_{[0,1]}(1) = \alpha - 1 = s|_{[1,2]}(1) = 1 + \sigma + \rho\,,$$
$$\frac{ds|_{[0,1]}}{dt}(1) = 3\alpha = \frac{ds|_{[1,2]}}{dt}(1) = -3 + 2\sigma + \rho\,,$$
$$\frac{d^2s|_{[0,1]}}{dt^2}(1) = 6\alpha = \frac{d^2s|_{[1,2]}}{dt^2}(1) = -6 + 2\sigma\,.$$

We arrive at the following linear system of equations for the unknown coefficients

$$\begin{bmatrix} 1 & -1 & -1 \\ 3 & -2 & -1 \\ 6 & -2 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \sigma \\ \rho \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -6 \end{bmatrix} \Rightarrow \begin{array}{c} \alpha = 2\,, \\ \sigma = 9\,, \\ \rho = -9\,. \end{array}$$

▲

(0-1.b) ⚃ (6 pts.) For parameters $\alpha, \beta, \gamma, \delta, \sigma, \rho \in \mathbb{R}$ we define the function

$$s(t) = \begin{cases} 2 & \text{for} \quad -1 \leq t < 0\,, \\ \sigma t^3 + \rho t^2 + \gamma t + \delta & \text{for} \quad 0 \leq t < 1\,, \\ -t^3 + \alpha t^2 + \beta t - 1 & \text{for} \quad 1 \leq t \leq 2\,. \end{cases}$$

Determine the parameters $\alpha, \beta, \gamma, \delta, \sigma, \rho \in \mathbb{R}$ such that s is a cubic spline with respect to the knot set $\mathcal{M} := \{-1, 0, 1, 2\}$.

$\alpha =$ ☐ , $\beta =$ ☐ $\gamma =$ ☐ ,

$\delta =$ ☐ , $\sigma =$ ☐ $\rho =$ ☐ .

SOLUTION of (0-1.b):

- At $t = 0$: $\delta = 2, \gamma = 0, \rho = 0$.
- At $t = 1$: From the continuity conditions

$$\begin{array}{rcl} \sigma + 2 & = & -1 + \alpha + \beta - 1 \\ 3\sigma & = & -3 + 2\alpha + \beta \\ 6\sigma & = & -6 + 2\alpha \end{array} \blacktriangleright \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 2 & 0 & -6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \sigma \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} \Rightarrow \begin{array}{c} \alpha = -9\,, \\ \beta = 9\,, \\ \sigma = -4\,. \end{array}$$

▲

(0-1.c) ⚃ (6 pts.) For parameters $\alpha, \beta, \gamma, \delta, \sigma, \rho \in \mathbb{R}$ we define the function

$$s(t) = \begin{cases} 1 & \text{for} \quad -1 \leq t < 0\,, \\ \alpha t^3 + \beta t^2 + \gamma t + \delta & \text{for} \quad 0 \leq t < 2\,, \\ -t^3 + \rho t^2 + \sigma t - 7 & \text{for} \quad 2 \leq t \leq 3\,. \end{cases}$$

Determine the parameters $\alpha, \beta, \gamma, \delta, \sigma, \rho \in \mathbb{R}$ such that s is a cubic spline with respect to the knot set $\mathcal{M} := \{-1, 0, 2, 3\}$.

$\alpha =$ [ ] , $\beta =$ [ ] $\gamma =$ [ ] ,

$\delta =$ [ ] , $\sigma =$ [ ] $\rho =$ [ ] .

SOLUTION of (0-1.c):

- At $t = 0$: $\delta = 1, \gamma = 0, \beta = 0$
- At $t = 2$:

$$\begin{aligned} 8\alpha + 1 &= -8 + 4\rho + 2\sigma - 7 \\ 12\alpha &= -12 + 4\rho + \sigma \\ 12\alpha &= -12 + 2\rho \end{aligned} \quad \blacktriangleright \quad \begin{bmatrix} 8 & -4 & -2 \\ 12 & -4 & -1 \\ 12 & -2 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \rho \\ \sigma \end{bmatrix} = \begin{bmatrix} -16 \\ -12 \\ -12 \end{bmatrix} \Rightarrow \begin{aligned} \alpha &= -2\,, \\ \rho &= -6\,, \\ \sigma &= 12\,. \end{aligned}$$

▲

End Problem 0-1 , 18 pts.

Problem 0-2: Monomial representation of Chebychev polynomials

Chebychev polynomials are important both for theoretical analysis and algorithm design. For isntance, as we have seen in [Lecture → Section 6.2.3.3] Chebyshev interpolants should internally be represented through their Chebychev expansion. In this problem we will study an algorithm for converting Chebychev expansions into monomial expansions.

This problem relies about elementary facts about Chebychev polynomials as introduced in [Lecture → Section 6.2.3.1].

On $[-1,1]$ the n-th Chebychev polynomial T_n, $n \in \mathbb{N}_0$ is defined as [Lecture → Def. 6.2.3.3]

$$T_n(t) := \cos(n \arccos t) \qquad -1 \leq t \leq 1 \,. \tag{0.2.1}$$

That T_n is a polynomial of degree $\leq n$ is clear from the 3-term recursion

$$T_{n+1}(t) = 2t\,T_n(t) - T_{n-1}(t) \quad , \quad T_0 \equiv 1 \,, \quad T_1(t) = t \,, \quad n \in \mathbb{N} \,. \qquad \text{[Lecture → Eq. (6.2.3.5)]}$$

The set $\{T_0, \ldots, T_n\}$ forms a basis of $\mathcal{P}_n$.

(0-2.a) ⚃ (8 pts.) Below is an incomplete implementation of a C++ function

```
Eigen::VectorXd chebexpToMonom(const Eigen::VectorXd &a);
```

that expects the coefficients $a_j \in \mathbb{R}$, $j = 0, \ldots, n$, of the Chebyshev expansion

$$p(t) = \sum_{j=0}^{n} a_j T_j(t) \,, \quad t \in \mathbb{R} \,,$$

of a polynomial $p \in \mathcal{P}_n$ to be passed in the vector `a` and returns the coefficients r_j, $j = 0, \ldots, n$ with respect to the monomial basis,

$$p(t) = r_0 + r_1 t + r_2 t^2 + \cdots + r_n t^n \,,$$

collected in a vector $[r_0, r_1, \ldots, r_n]^\top$.

Fill in the missing pieces of the following code, which relies on [Lecture → Eq. (6.2.3.5)].

```
Eigen::VectorXd chebexpToMonom(const Eigen::VectorXd &a) {
  const int n = a.size() -1; // degree of polynomial
  assert(n >= 0);
  Eigen::VectorXd r{ Eigen::VectorXd::Zero(n+1) };
  r[0] = [ A ] ;
  if (n > 0) {
    r[1] = [ B ] ;
    if (n > 1) {
      // two vectors temporarily storing the monomial coefficients
      // of Chebychev polynomials of two consecutive degrees
      std::array<Eigen::VectorXd, 2> c
       { Eigen::VectorXd::Zero(n+1), Eigen::VectorXd::Zero(n+1) };
      c[0][0] = 1.0; c[1][1] = 1.0;
      for (int j=2; j<=[ C ] ; ++j) {
            c[j%2][0] *= -1.0;
```

```
                for (int k= [D] ; k <= j; ++k)
                  c[j%2][k] = [E] *c[ [F] ][ [G] ] - c[j%2][k];
                r += [H] *c[j%2];
            }
        }
    }
    return r;
}
```

- [A] ≙ [ ]
- [B] ≙ [ ]
- [C] ≙ [ ]
- [D] ≙ [ ]
- [E] ≙ [ ]
- [F] ≙ [ ]
- [G] ≙ [ ]
- [H] ≙ [ ]

SOLUTION of (0-2.a):

C++ code 0.2.2: Implementation of a function providing the monomial coefficients of a Chebychev expansion

```cpp
Eigen::VectorXd chebexpToMonom(const Eigen::VectorXd &a) {
  const int n = a.size() -1; // degree of polynomial
  assert(n >= 0);
  // Vector for returning monomial coefficients
  Eigen::VectorXd r{ Eigen::VectorXd::Zero(n+1) };
  r[0] = a[0]; // T_0 ≡ 1
  if (n > 0) {
    r[1] = a[1];  // T_1(t) = t
    if (n > 1) {
      // two vectors temporarily storing the monomial coefficients
      // of Chebychev polynomials of two consecutive degrees
      std::array<Eigen::VectorXd, 2> c
        { Eigen::VectorXd::Zero(n+1), Eigen::VectorXd::Zero(n+1) };
      c[0][0] = 1.0; c[1][1] = 1.0; // For T_0 and T_1
      // Run through all Chebychev polynomials
      for (int j=2; j<=n; ++j) {
        // Update monomial coefficients by 3-term recursion
        c[j%2][0] *= -1.0;
        for (int k=1; k <= j; ++k) {
          // Formula for 3-term recursion
          c[j%2][k] = 2*c[(j+1)%2][k-1] - c[j%2][k];
        }
        // Add monomial coefficients of contribution of T_j
        r += a[j]*c[j%2];
      }}}
  return r;
}
```

▲

End Problem 0-2 , 8 pts.

Problem 0-3: Finding domains of analyticity

The developments of [Lecture → Lemma 6.2.2.53], [Lecture → Rem. 6.2.3.26], and [Lecture → Section 6.5.3] make clear that knowledge about that maximal subdomain $D \subset \mathbb{C}$ of the complex plane to which an analytic function can be extended from its original real interval of definition is key to predicting the speed of exponential convergence of approximation schemes like Chebychev interpolation and trigonometric interpolation. In this problem we discuss the analytic extension of 1-periodic functions.

Depends on [Lecture → Rem. 6.2.2.59] and is related to [Lecture → § 6.5.3.16].

In order to analyze the trigonometric functions occurring in this problem, the following identities can be useful:

$$\begin{aligned} \sin(x+\imath y) &= \sin(x)\cosh(y) + \imath\cos(x)\sinh(y) \quad \forall x,y \in \mathbb{R}\,, \\ \cos(x+\imath y) &= \cos(x)\cosh(y) - \imath\sin(x)\sinh(y) \quad \forall x,y \in \mathbb{R}\,, \\ \sin^2(z) + \cos^2(z) &= 1 \quad \forall z \in \mathbb{Z}\,. \end{aligned} \tag{0.3.1}$$

The following result gives domains of analyticity of special functions:

Theorem 0.3.2. domains of analyticity of special functions

- *Every polynomial is analytic on* $\mathbb{C}$.
- *Every rational function, that is, a quotient of two polynomials, is analytic in the complement of the set of zeros of its denominator.*
- *The functions* $\exp$, $\sin$, *and* $\cos$ *are analytic on* $\mathbb{C}$.
- *The functions* $\log : \mathbb{R}^+ \to \mathbb{R}$ *and* $\sqrt{\cdot} : \mathbb{R}_0^+ \to \mathbb{R}$ *can be extended to analytic functions on* $\mathbb{C} \setminus \mathbb{R}_0^-$.

(0-3.a) ⚃ (6 pts.) We consider the 1-periodic function

$$f(t) = b\log(a + \sin(2\pi t))\,, \quad t \in \mathbb{R}\,, \quad b > 0,\ a > 1\,. \tag{0.3.3}$$

Determine the largest possible subset D of $\mathbb{C}$ to which f can be extended analytically.

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \mathrm{Re}(z) \in \boxed{}\,, \\ \mathrm{Im}(z) \in \boxed{} \end{array} \right\}.$$

SOLUTION of (0-3.a):

We write $P := \mathbb{C} \setminus D \subset \mathbb{C}$ for the complement of the domain of analyticity (the "domain of singularity") of f. Multiplication with b does not have any impact on analyticity. By Thm. 0.3.2 we have

$$\begin{aligned} P &= \{z \in \mathbb{C} :\ a + \sin(2\pi z) \in \mathbb{R}_0^-\} \\ &= \{z \in \mathbb{C} :\ \mathrm{Im}(a + \sin(2\pi z)) = 0 \quad \wedge \quad \mathrm{Re}(a + \sin(2\pi z)) \le 0\}\,. \end{aligned}$$

Next, we use (0.3.1) with $z = x + \imath y$, $x, y \in \mathbb{R}$:

$$\sin(2\pi z) = \sin(2\pi x)\cosh(2\pi y) + \imath\cos(2\pi x)\sinh(2\pi y)\,,$$

which means for $z = x + \imath y$

$$\mathrm{Im}(a + \sin(2\pi z)) = \cos(2\pi x)\sinh(2\pi y) \quad , \quad \mathrm{Re}(a + \sin(2\pi z)) = a + \sin(2\pi x)\cosh(2\pi y) \; .$$

We first examine $\mathrm{Im}(a + \sin(2\pi z)) = 0 \Leftrightarrow \cos(2\pi x)\sinh(2\pi y) = 0$:

$$\mathrm{Im}(a + \sin(2\pi z)) = 0 \quad \Leftrightarrow \quad y = 0 \quad \vee \quad x \in \tfrac{1}{2}\mathbb{Z} + \tfrac{1}{4} \; .$$

The case $y = 0$ need not be considered any further, because, since $a > 1$,

$$y = 0 \quad \Rightarrow \quad \mathrm{Re}(a + \sin(2\pi z)) = a + \sin(2\pi x) > 0 \; .$$

The other case $x \in \frac{1}{2}\mathbb{Z} + \frac{1}{4}$ means

$$x = (2k+1)\tfrac{1}{4} \; , \quad k \in \mathbb{Z} \; , \quad \Rightarrow \quad \sin(2\pi x) = (-1)^k \; .$$

Hence, $\mathrm{Re}(a + \sin(2\pi z)) \leq 0$ is possible only if

$$x = (4k+3)\frac{1}{4} \; , \quad k \in \mathbb{Z} \; , \quad a - \cosh(2\pi y) \leq 0 \quad \Leftrightarrow \quad |y| \geq \frac{1}{2\pi}\cosh^{-1}(a) \; .$$

Summing up we have found

$$D = \mathbb{C} \setminus \{z \in \mathbb{C} : \; \mathrm{Re}(z) \in \mathbb{Z} + \tfrac{3}{4} \, , \, |\mathrm{Im}(z)| \geq \zeta, \; \zeta > 0, \; \cosh(2\pi\zeta) = a\} \; .$$

The presence of the parameter b has no impact on the domain of analyticity.

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \mathrm{Re}(z) \in \mathbb{Z} + \frac{3}{4} \, , \\ \mathrm{Im}(z) \in \{y \in \mathbb{R} : \; |y| \geq q, \; q > 0, \; \cosh(2\pi q) = a\} \end{array} \right\} \; .$$

▲

(0-3.b) ⚃ (6 pts.) We consider the 1-periodic function

$$f(t) = a \log(b - \cos(2\pi t)) \; , \quad t \in \mathbb{R} \; , \quad b > 1, \; a > 1 \; . \qquad (0.3.4)$$

Determine the largest possible subset D of $\mathbb{C}$ to which f can be extended analytically.

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \mathrm{Re}(z) \in \boxed{} \, , \\ \mathrm{Im}(z) \in \boxed{} \end{array} \right\} \; .$$

SOLUTION of (0-3.b):

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \mathrm{Re}(z) \in \mathbb{Z} \, , \\ \mathrm{Im}(z) \in \{y \in \mathbb{R} : \; |y| \geq q, \; \cosh(2\pi q) = b\} \end{array} \right\} \; .$$

▲

(0-3.c) (6 pts.) We consider the 1-periodic function

$$f(t) = a\sqrt{b - \sin(2\pi t)}\ , \quad t \in \mathbb{R}\ , \quad b > 1,\ a > 1\ . \tag{0.3.5}$$

Determine the largest possible subset D of $\mathbb{C}$ to which f can be extended analytically.

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \mathrm{Re}(z) \in \boxed{}\ , \\ \mathrm{Im}(z) \in \boxed{} \end{array} \right\} .$$

SOLUTION of (0-3.c):

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{l} \mathrm{Re}(z) \in \mathbb{Z} + \frac{1}{4}\ , \\ \mathrm{Im}(z) \in \{y \in \mathbb{R} : \ |y| \geq q,\ \cosh(2\pi q) = b\} \end{array} \right\} .$$

▲

End Problem 0-3 , 18 pts.